

# Brief Introduction to Random Matrices: Dlya Balbesov :-))

(Dated: June 1, 2005)

In many physical problems, especially in nuclear and condensed matter physics, the quantum mechanical Hamiltonian, which models the underlying physical complexity, involves a random matrix belonging to one of the three standard Gaussian ensembles: the Gaussian Unitary Ensemble (GUE), the Gaussian Orthogonal Ensemble (GOE) or the Gaussian Symplectic Ensemble (GSE). The appropriate choice is dictated by the symmetries of the theory. In condensed matter physics such problems arise in the study of the electron localization phenomena in disordered conductors and semi-conductors. In the mesoscopic regime, applications include universal conductance fluctuations, Aronov-Altshuler oscillations and persistent currents. These are weak-localization effects. In the strongly localized domain, problems such as long disordered quantum wires and the integer quantum Hall effect are amenable to a random-matrix (RM) formulation. In the ballistic regime, where disorder is negligible, classical chaos in the dynamics of electron scattering gives rise to the complexity that can be described within RM theory. Thus, another recent important application has been the study of electron transport across microstructures, constructed in the shape of classically chaotic billiards in the presence of external magnetic field. This allows one to investigate the continuous crossover between two random matrix ensembles  $GOE \rightarrow GUE$  as probed by the mean conductance and its correlations for increasing magnetic field. Also connection between the Shutherland-Calogero model and random matrix ensembles have been established and discussed in the literature. Following notes are brief introduction to the field of random matrices.

## PRELIMINARIES

According to quantum mechanics, the energy levels of a system are supposed to be described by the eigenvalues of a Hermitian operator, called the Hamiltonian. The energy-level scheme of a system consists in general of a continuum and a certain, perhaps a large, number of discrete levels which will be the main interest of our study. Choosing the basis in the Hilbert space we represent our Hamiltonians by finite (or in principle infinite) dimensional matrices. If we can solve the eigenvalue equation

$$\hat{H}\psi_i = E_i\psi_i \quad (1)$$

we shall get all the eigenvalues and eigenfunctions of the system, and any physical information can then be deduced, in principle, from this knowledge. But in practice in the case of the nucleus for example or small metallic particle of random shape there are two difficulties:

- first, we do not know the Hamiltonian and,
- second, even if we did, it would be far too complicated to attempt to solve it.

Therefore for the very beginning we shall be making statistical hypotheses on the Hamiltonian  $\hat{H}$  compatible with the general symmetry properties. Choosing a complete set of functions as basis, we represent the Hamiltonian operators as matrices. The elements of these matrices are random variables whose distributions are restricted only by the general symmetry properties we might impose on the ensemble of operators. Statistical

theory doesn't predict the detailed level sequence of any one nucleus (or energy levels in each particular metallic granule) but it does describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus (granule) which is to complicated to be understood in detail.

In classical mechanics a system may be in one of the many possible states, but one does not ask in which particular state a given system is. As in orthodox statistical mechanics we shall consider an ensemble of Hamiltonians (matrices) each of which could describe a different nucleus (granule). There is a reasonable expectation that a system under observation will be described by ensemble average. This expectation is strong because the system might be one of huge variety of systems and very few of them will deviate much from a properly chosen ensemble average. Wigner was the first to propose hypothesis that the local statistical behavior of levels in a simple sequence is identical with the eigenvalues of a random matrix. A simple sequence is one whose levels all have the same spin, parity, and other strictly conserved quantities, if any, which result from the symmetry of the system.

The physical properties of metals depend characteristically on their excitation spectra. In bulk metal at high temperatures the electronic energy levels lie very near to one another and are broad enough to overlap and form a continuous spectrum. As the sample gets smaller, this spectrum becomes discrete, and as the temperature decrease the width of the individual levels decrease. If the metallic particles are minute enough and at low temperatures, the spacing of the electronic en-

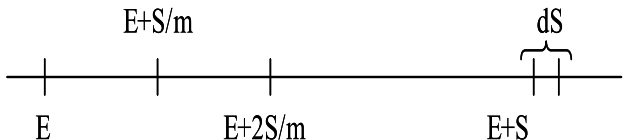


FIG. 1: Schematic diagram for  $P(t)$  calculation.

ergy levels  $\Delta$  may eventually become much larger than the other energies, such that level width  $\Gamma$  and thermal energy  $T$ :  $\Delta \gg \Gamma$  and  $\Delta \gg T$ . Under such conditions the thermal and the electromagnetic properties of the fine metallic particles may deviate considerably from those of the bulk metal. In order to find conditions when  $\Delta \gg T$  holds consider a small metallic particles at low temperatures. The number of electrons in a volume  $V$  is  $N = 4\pi p_f^3 V / 3h^3$ . The energy of an excitation near Fermi surface is  $E_f = p_f^2 / 2m^*$  where  $m^*$  some effective mass of an electron. The level density at zero excitation is therefore  $\nu = dN/dE_f = 4\pi p_f V m^* / h^3$  and the average level spacing  $\Delta$  is the inverse of this quantity  $\Delta = \nu^{-1}$ . For example, a metallic particle of size  $10^6 - 10^{-7} cm$  contains  $10^4 - 10^5$  electrons and at  $T \sim 10K = 10^{-3} eV$ ,  $\Delta \sim 1eV \gg T$ .

### LEVEL SPACINGS

Let assume that level density as a function of the excitation energy as known and consider an interval of energy  $\delta E$  centered at  $E$ . This interval is much smaller compared with  $E$ , whereas it is large enough to contain many levels  $E \gg \delta E \gg \Delta$  again here  $\Delta$  is mean distance between neighboring levels. How are the levels are distributed in this interval? Although the level density varies for sample to sample (by saying sample we assume nucleus or small metallic particle or in principle any other physical system) the fluctuations in the precise positions of the levels seem not to depend on the sample. As the density of levels is nearly constant in this interval we may might think that they occur at random positions without regard to one another, the only condition being that their density be a given constant. However such is not the case. It is true that nuclear levels with different spin and parity or atomic levels with different sets of good quantum numbers seem to have no influence on each other. However, levels with the same set of good quantum numbers show a large degree of regularity, for example they rarely occur close together. To distinguish between various possibilities we define the distribution function of level spacing. Let  $E_1, E_2, \dots, E_n$  be the position of the successive levels in the interval  $\delta E$  ( $E_1 < E_2 < \dots < E_n$ ) and

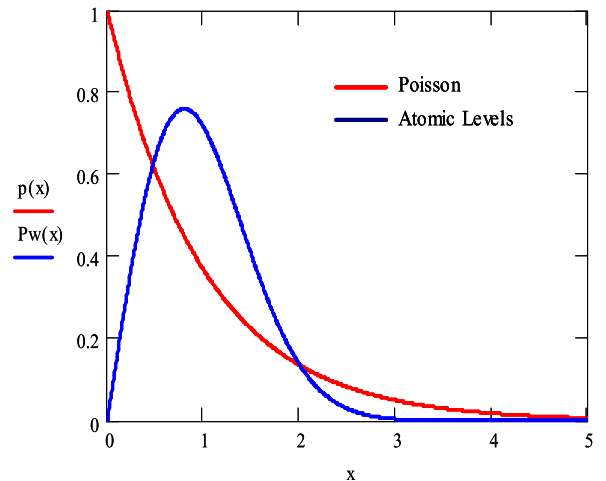


FIG. 2: Plot of the Poisson distribution function and the density of spacings between odd parity atomic levels of a group of elements in the region of osmium. The levels in each element were separated according to the angular momentum.

let  $S_1, S_2, \dots$  be their distances apart  $S_i = E_{i+1} - E_i$ . The average value of  $S_i$  is the mean spacing  $\Delta$ . We also define relative spacings  $t_i = S_i / \Delta$ . The probability density function  $P(t)$  is defined by condition that  $P(t)dt$  is the probability that any  $t_i$  will have value between  $t$  and  $t + dt$ .

For the simplest case in which the positions of the energy levels are not correlated the probability that any  $E_i$  will fall between  $E$  and  $E + dE$  is independent of  $E$  and is simply  $\rho dE$  where  $\rho = \Delta^{-1}$  is the average number of levels in the unit interval of energy. Let us determine the probability of spacing  $S$ , that is, given a level at  $E$ , what is the probability of having no level in the interval  $(E, E + S)$  and one level in the  $dS$  at  $E + S$ . For this we divide the interval  $S$  into  $m$  equal parts Fig.(1). Because the levels are independent, the probability of having no level in  $(E, E + S)$  is the product of probabilities of having no level in any of these  $m$  parts. If  $m$  is large, so that  $S/m$  is small then we can write this as

$$\lim_{m \rightarrow \infty} \left( 1 - \rho \frac{S}{m} \right)^m = \exp(-\rho S). \quad (2)$$

Moreover, the probability of having a level in  $dS$  at  $E + S$  is  $\rho dS$ , therefore, given level at  $E$ , the probability that there is no level in  $(E, E + S)$  and having one level in  $dS$  at  $E + S$  is

$$\exp(-\rho S) \rho dS \quad (3)$$

or in terms of variable  $t = S/\Delta = \rho S$

$$P(t)dt = \exp(-t)dt. \quad (4)$$

This is known as the Poisson distribution or the spacing rule for random levels.

That Eq.(4) is not correct for nuclear levels of the same spin and parity or for atomic levels of the same parity and orbital and spin angular momenta is clearly seen by a comparison Fig.(2).

When the experimental situation was not yet conclusive Winger proposed the following rules for spacing distributions:

1. In the sequence of levels with the same spin and parity, called a *simple sequence*, the probability density function for spacing is given by

$$P_W(t) = \frac{\pi}{2}t \exp\left(-\frac{\pi}{4}t^2\right). \quad (5)$$

2. The levels with different spin and parity are not correlated.

Two simple arguments give rise to Rule 1. It is reasonable to expect that, given a level at  $E$ , the probability that another level will lie around  $E + S$  is proportional to  $S$  for very small  $S$ . (**Note:** from physical point of view such an expectation means level repulsion but assumption that probability  $P(S)$  is linearly proportional to  $S$  actually follows form nowhere and in principle it may be proportional to  $S$  at some power  $S^\beta$  which actually happens for Hamiltonians of some symmetry classes. Linear proportionality is the simplest assumption and turns out to be correct for systems with time reversal symmetry and even spin which correspond to GOE).

Let at this point define the  $n$ -point correlation function  $P_n(E_1, E_2, \dots, E_n)$  so that  $P_n dE_1 dE_2 \dots dE_n$  is the probability function of finding a level in each of the intervals  $(E_1, E_1 + dE_1), \dots, (E_n, E_n + dE_n)$ . The two simple arguments of Wigner given in the derivation of Rule 1 are equivalent to the following: the two-point correlation function  $R_2(E_1, E_2)$  is linear in the variance  $|E_1 - E_2|$  and three and higher order correlation functions are negligibly small.

## TARGET QUANTITIES

In the mathematical model our systems are characterized by their Hamiltonians, which in turn are represented by Hermitian matrices. Let us look into the structure of these matrices. The low-lying energy levels (eigenvalues) are far apart and each may be described by different set of quantum numbers. As we go to higher excitations, the levels draw closer, and because of their mutual interference most of the approximate quantum numbers lose their usefulness, for they are no longer exact. At still higher excitations the interference is so great that some quantum numbers may become entirely meaningless. However, there may be certain exact integrals of

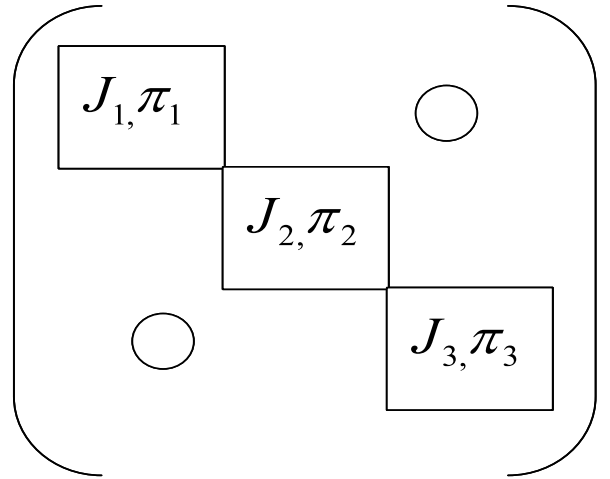


FIG. 3: Structure of the Hamiltonian.

motion, such as total spin or parity, and the quantum numbers corresponding to them are conserved whatever the excitation may be. If the basic functions are chosen to be the eigenfunctions of these conserved quantities, all Hamiltonian matrices of the ensemble will reduce to the form of diagonal blocks. One block will correspond uniquely to each set of exact quantum numbers. The matrix elements lying outside these blocks will all be zero, and levels belonging to two different blocks will be statistically uncorrelated Fig.(3). As to the levels corresponding to the same block, the interactions are so complex that any regularity resulting from partial diagonalization will be washed out.

We will assume that such a basis has already been chosen and restrict our attention to one of the diagonal blocks an  $N \times N$  Hermitian matrix in which  $N$  is a large but fixed positive integer. We will be interested in studying of following quantities: density of states  $\nu(E)$  namely - how many levels are in the given energy interval  $E$ , two point correlation function  $P_2(E_1, E_2)$  or equivalently density distribution function of level spacings  $P(t)$  and density-density correlation function  $R(E_1, E_2) = \langle \nu(E_1)\nu(E_2) \rangle$  where  $\langle \dots \rangle$  means statistical averaging over ensemble of random matrices.

## GAUSSIAN ENSEMBLES OF HERMITIAN MATRICES

### Definitions

Two special cases will be considered here: Gaussian Orthogonal Ensemble (GOE) and Gaussian Unitary Ensemble (GUE) (Gaussian Symplectic Ensemble we will leave aside as more complicated).

Consider now the even-spin quantum system. Every such system is invariant under time reversal transformation and with each system we will associate a real symmetric matrix  $\hat{H}$ .

**Definition 1:** The Gaussian Orthogonal Ensemble  $E_{1G}$  is defined in the space  $T_{1G}$  of real symmetric matrices by two requirements:

1. The ensemble is invariant under every transformation  $H \rightarrow OTO^{-1}$  of  $T_{1G}$  into itself, where  $O$  is any real orthogonal matrix ( $\det O = 1, O^{-1} = O^T$ ).
2. The various elements  $H_{kj}$ ,  $k \leq j$ , are statistically independent.

These requirements, expressed in the form of equations, read as follows

1. The probability  $P(H)dH$  that a system of  $E_{1G}$  will belong to the volume element  $dH = \prod_{k \leq j} dH_{kj}$  is invariant under real orthogonal transformations:  $P(H')dH' = P(H)dH$  where  $H' = OHO^{-1}$  and  $O^T O = OO^T = 1$ .
2. This probability density function  $P(H)$  is a product of functions, each of which depends on at most a single variable:

$$P(H) = \prod_{k \leq j} P_{kj}(H_{kj}). \quad (6)$$

Consider now a system without invariance under time reversal with any spin which belongs to Gaussian Unitary Ensemble. Such system are easily created in principle by putting an ordinary atom, nucleus or quantum dot into the externally generated magnetic field. The external field is not affected by time reversal operation. However, for the unitary ensemble to be applicable, the splitting of levels by the magnetic field must be at least as large as the average level spacing in the absence of the magnetic field. The magnetic field must be so strong that it will completely "mix up" the level structure that would exist in zero field; for otherwise our random hypothesis can't be justified. A system without time-reversal invariance has a Hamiltonian that may be an arbitrary Hermitian matrix not restricted to be real.

**Definition 2:** The Gaussian Unitary Ensemble  $E_{2G}$  is defined in the space  $T_{2G}$  of Hermitian matrices by the following properties

1. The probability  $P(H)dH$  that a system of  $E_{2G}$  will belong to the volume element  $dH = \prod_{k \leq j} dH_{kj}^{(0)} \prod_{k < j} dH_{kj}^{(1)}$ , where  $H_{kj}^{(0)}$  and  $H_{kj}^{(1)}$  are real and imaginary parts of  $H_{kj}$ , is invariant under every transformation  $H \rightarrow UHU^\dagger$  of  $T_{2G}$  into itself, where  $U$  is any unitary matrix.
2. Various linearly independent components of  $H$  are also statistically independent

In mathematical language these requirements are

1.  $P(H')dH' = P(H)dH$  where  $H' = UHU^\dagger$  with  $U$  being unitary matrix.
2.  $P(H)$  is a product of functions, each of which depends on a single variable:

$$P(H) = \prod_{k \leq j} P_{kj}^{(0)}(H_{kj}^{(0)}) \prod_{k < j} P_{kj}^{(1)}(H_{kj}^{(1)}). \quad (7)$$

### Joint Probability Density Function for Matrix Elements

Now we come to the question of the extent to which we are still free to specify the joint probability density function  $P(H)$ . It will be seen that the two postulates of invariance and statistical independence elaborated above fix uniquely the functional form of  $P(H)$ . Instead of general proof I would like to consider some example which develops our intuition and provides desirable result at the same time.

Let us now examine the consequences of the statistical independence of the various components of  $\hat{H}$  for the GOE. We will simplify our analysis greatly by considering random matrix  $2 \times 2$  instead of general case  $N \times N$  but it will be sufficiently enough for our purposes. Statistical independence of matrix elements of  $\hat{H}$  implies that joint probability distribution function is product of distribution functions for each element

$$P(H) = P_{11}(H_{11})P_{22}(H_{22})P_{12}(H_{12}). \quad (8)$$

An orthogonal transformation infinitesimally close to unity

$$O = \begin{pmatrix} 1 & -d\theta \\ d\theta & 1 \end{pmatrix} \quad (9)$$

changes  $H$  to  $H' = H + dH = OHO^{-1}$  such that

$$H' = \begin{pmatrix} 1 & -d\theta \\ d\theta & 1 \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} = \begin{pmatrix} H_{11} - 2H_{12}d\theta & H_{12} + (H_{11} - H_{22})d\theta \\ H_{12} + (H_{11} - H_{22})d\theta & H_{22} + 2H_{12}d\theta \end{pmatrix} \quad (10)$$

and we have

$$\begin{cases} dH_{11} = -dH_{22} = -2H_{12}d\theta \\ dH_{12} = (H_{11} - H_{22})d\theta \end{cases} \quad (11)$$

As soon as we require that probability density  $P(H)$  is invariant under the transformation  $O$  then its derivative with respect to  $\theta$  must vanish

$$\frac{\partial P}{\partial \theta} = 0 \Rightarrow \frac{\partial P_{11}}{\partial \theta} P_{22} P_{12} + \frac{\partial P_{22}}{\partial \theta} P_{11} P_{12} + \frac{\partial P_{12}}{\partial \theta} P_{11} P_{22} = 0 \quad (12)$$

Dividing everything by the product  $P_{11}P_{22}P_{12}$  and assuming that

$$\frac{\partial P_{ik}}{\partial \theta} = \frac{dP_{ik}}{dH_{ik}} \frac{dH_{ik}}{d\theta} \quad (13)$$

we arrive to relation

$$\frac{1}{P_{11}} \frac{dP_{11}}{dH_{11}} \frac{dH_{11}}{d\theta} + \frac{1}{P_{22}} \frac{dP_{22}}{dH_{22}} \frac{dH_{22}}{d\theta} + \frac{1}{P_{12}} \frac{dP_{12}}{dH_{12}} \frac{dH_{12}}{d\theta} = 0. \quad (14)$$

Now calculating derivatives  $dH_{ik}/d\theta$  from Eq.(11)

$$\frac{dH_{11}}{d\theta} = -\frac{dH_{22}}{d\theta} = -2H_{12}, \quad \frac{dH_{12}}{d\theta} = H_{11} - H_{22} \quad (15)$$

writing  $\frac{1}{P_{ik}} \frac{dP_{ik}}{dH_{ik}} = \frac{d \ln P_{ik}}{dH_{ik}}$  and after small rearrangements we have

$$\frac{2}{H_{11} - H_{22}} \left( \frac{d \ln P_{22}}{dH_{22}} - \frac{d \ln P_{11}}{dH_{11}} \right) = -\frac{1}{H_{12}} \frac{d \ln P_{12}}{dH_{12}} \quad (16)$$

Right hand side of the Eq.(16) depends only on  $H_{12}$  and left hand side on  $H_{11}$  and  $H_{22}$  and do not depend on  $H_{12}$ . The only possibility to have such equality is that whole equation is just a some constant  $2a$ , which gives us two separate differential equations:

$$\begin{cases} -\frac{1}{H_{12}} \frac{d \ln P_{12}}{dH_{12}} = 2a, \\ \frac{1}{H_{11} - H_{22}} \left( \frac{d \ln P_{22}}{dH_{22}} - \frac{d \ln P_{11}}{dH_{11}} \right) = a \end{cases} \quad (17)$$

From the first equation we immediately have that

$$P_{12} \propto \exp(-aH_{12}^2). \quad (18)$$

Then just by looking on the second equation and trying solution in the form  $\ln P_{11} = -\frac{a}{2}H_{11}^2$  and  $\ln P_{22} = -\frac{a}{2}H_{22}^2$  we see that it indeed works. Combining all together we have joint density distribution function

$$P(H) \propto \exp \left[ -\frac{a}{2}(H_{11}^2 + 2H_{12}^2 + H_{22}^2) \right]. \quad (19)$$

Last formula may be written in more symmetric way. Lets calculate square of our matrix  $H$

$$H^2 = \begin{pmatrix} H_{11}^2 + H_{12}^2 & H_{12}(H_{11} + H_{22}) \\ H_{12}(H_{11} + H_{22}) & H_{12}^2 + H_{22}^2 \end{pmatrix} \quad (20)$$

and then take trace of  $H^2$

$$\text{Tr} H^2 = H_{11}^2 + 2H_{12}^2 + H_{22}^2 \quad (21)$$

which is precisely power of the exponent in Eq.(19). Last observation makes possible to write joint probability density function in the general form without regards to some specific structure of the Hamiltonian

$$P(H) \propto \exp \left[ -\frac{a}{2} \text{Tr} H^2 \right]. \quad (22)$$

Of course we got this result in the oversimplified situation  $2 \times 2$  symmetric real matrix. One may go ahead and repeat this derivation in the general case  $N \times N$  matrix which belongs to either of three ensembles and arrive to the following theorem: *In all the above cases the form of  $P(H)$  is automatically restricted to*

$$P(H) \propto \exp(-A \text{Tr} H^2 + B \text{Tr} H + C) \quad (23)$$

where  $A$  is real and positive and  $B$  and  $C$  are real.

### Integration measure

The joint density distribution function Eq.(22) is normalized as  $\int dH P(H) = 1$  but we must specify the integration measure  $dH$ . It is simply the product of differentials of the independent matrix elements

$$dH = \left( \prod_i H_{ii} \right) \times \begin{cases} \prod_{i < j} dH_{ij} & \text{GOE} \\ \prod_{i < j} d\text{Re} H_{ij} d\text{Im} H_{ij} & \text{GUE} \end{cases} \quad (24)$$

and that measure is also invariant under orthogonal and unitary transformations respectively. Of course we are the most interested in extracting distributions for the eigenvalues of  $\hat{H}$  from the joint density Eq.(22) of matrix elements. Thus we just need to invoke the orthogonal or unitary transformation diagonalizing  $\hat{H}$  as

$$H = \begin{cases} O \text{diag}(E_1, \dots, E_N) O^{-1} & \text{GOE} \\ U \text{diag}(E_1, \dots, E_N) U^{-1} & \text{GUE} \end{cases} \quad (25)$$

The structure of Eq.(22) immediately provides as  $P(H) \propto \exp(-A \sum_i E_i^2)$ . To get the reminder we need to re-express the measure  $dH$  in terms of the  $dE$  and "angles" in the diagonalizing matrices

$$dH = \left( \prod_i dE_i \right) \left( \prod \text{dangles} \right) J \quad (26)$$

$$J = \det \left| \frac{\partial(H_{11}, H_{12}, \dots, H_{NN})}{\partial(E_1, \dots, E_N, \theta, \varphi, \dots)} \right| \quad (27)$$

Again instead of carrying out rigorous derivations for the general cases we will work out the simple calculation for  $2 \times 2$  matrices which will provide all desirable results.

**1) Measure J for GOE:** the orthogonal transformation by the  $2 \times 2$  matrix is a simple rotation

$$O = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (28)$$

which gives a relation between eigenvalues+angle  $E_1, E_2, \theta$  and matrix elements of  $H$

$$H = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \quad (29)$$

$$= \begin{pmatrix} E_1 \cos^2 \theta + E_2 \sin^2 \theta & (E_1 - E_2) \cos \theta \sin \theta \\ (E_1 - E_2) \cos \theta \sin \theta & E_2 \cos^2 \theta + E_1 \sin^2 \theta \end{pmatrix} = \left| \det \begin{pmatrix} \cos^2 \theta & \sin^2 \theta & -(E_1 - E_2) \sin 2\theta \\ \frac{1}{2} \sin 2\theta & -\frac{1}{2} \sin 2\theta & (E_1 - E_2) \cos 2\theta \\ \sin^2 \theta & \cos^2 \theta & (E_1 - E_2) \sin 2\theta \end{pmatrix} \right|$$

By the definition of the Jacobian we have

$$J = \left| \det \begin{pmatrix} \frac{\partial H_{11}}{\partial E_1} & \frac{\partial H_{11}}{\partial E_2} & \frac{\partial H_{11}}{\partial \theta} \\ \frac{\partial H_{12}}{\partial E_1} & \frac{\partial H_{12}}{\partial E_2} & \frac{\partial H_{12}}{\partial \theta} \\ \frac{\partial H_{22}}{\partial E_1} & \frac{\partial H_{22}}{\partial E_2} & \frac{\partial H_{22}}{\partial \theta} \end{pmatrix} \right| = \quad (30)$$

$$J = \cos^2 \theta \left| \begin{array}{cc} -\frac{1}{2} \sin 2\theta & (E_1 - E_2) \cos 2\theta \\ \cos^2 & (E_1 - E_2) \sin 2\theta \end{array} \right| - \frac{1}{2} \sin 2\theta \left| \begin{array}{cc} \sin^2 \theta & -(E_1 - E_2) \sin 2\theta \\ \cos^2 & (E_1 - E_2) \sin 2\theta \end{array} \right| + \sin^2 \theta \left| \begin{array}{cc} \sin^2 \theta & -(E_1 - E_2) \sin 2\theta \\ -\frac{1}{2} \sin 2\theta & (E_1 - E_2) \cos 2\theta \end{array} \right| = \quad (31)$$

$$= |E_1 - E_2| \underbrace{(\sin^2 2\theta \cos^2 \theta + \sin^2 2\theta \sin^2 \theta)}_{\sin^2 2\theta} + \cos^4 \theta \cos 2\theta - \sin^4 \theta \cos 2\theta = |E_1 - E_2| (\sin^2 2\theta + \cos 2\theta (\cos^4 \theta - \sin^4 \theta)) =$$

$$= |E_1 - E_2| (4 \sin^2 \theta \cos^2 \theta + (\cos^2 \theta - \sin^2 \theta)^2) = |E_1 - E_2| (\cos^2 \theta + \sin^2 \theta)^2 = |E_1 - E_2|$$

The final result for desired Jacobian is

$$J_{\text{GOE}} = |E_1 - E_2| \quad (32)$$

**2) Measure J for GUE:** the unitary transformation by the is given by a  $2 \times 2$  matrix

$$U = \begin{pmatrix} \cos \theta & -e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & \cos \theta \end{pmatrix} \quad (33)$$

which gives a relation between eigenvalues+angles  $E_1, E_2, \theta, \varphi$  and matrix elements of  $H$ :

$$H = \begin{pmatrix} E_1 \cos^2 \theta + E_2 \sin^2 \theta & \frac{1}{2}(E_1 - E_2)e^{-i\varphi} \sin 2\theta \\ \frac{1}{2}(E_1 - E_2)e^{-i\varphi} \sin 2\theta & E_2 \cos^2 \theta + E_1 \sin^2 \theta \end{pmatrix}. \quad (34)$$

Setting up the he Jacobian we have

$$J = \left| \det \begin{pmatrix} \cos^2 \theta & \sin^2 \theta & -(E_1 - E_2) \sin 2\theta & 0 \\ \frac{1}{2}e^{-i\varphi} \sin 2\theta & -\frac{1}{2}e^{-i\varphi} \sin 2\theta & (E_1 - E_2)e^{-i\varphi} \cos 2\theta & -\frac{i}{2}(E_1 - E_2)e^{-i\varphi} \sin 2\theta \\ \frac{1}{2}e^{i\varphi} \sin 2\theta & -\frac{1}{2}e^{i\varphi} \sin 2\theta & (E_1 - E_2)e^{i\varphi} \cos 2\theta & \frac{i}{2}(E_1 - E_2)e^{i\varphi} \sin 2\theta \\ \sin^2 \theta & \cos^2 \theta & (E_1 - E_2) \sin 2\theta & 0 \end{pmatrix} \right| = \left| \det \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} \right| \quad (35)$$

and in stead of direct calculation of the determinant for  $4 \times 4$  matrix it is easier to reduce it to calculation of Shur's determinant of  $2 \times 2$  matrices

$$\det \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \det(\hat{A} - \hat{B}\hat{D}^{-1}\hat{C})\det\hat{D} \quad (36)$$

providing

$$J_{\text{GUE}} = (E_1 - E_2)^2 \cos \theta \sin \theta. \quad (37)$$

Of course we can extend results Eq.(32) and Eq.(37) to arbitrary dimensionality  $N$  of the Hamiltonian  $H$  and we will get

$$J = \left( \prod_{i < j}^N |E_i - E_j|^\beta \right) \times j(\text{angles}), \beta = \begin{cases} 1 & \text{GOE} \\ 2 & \text{GUE} \end{cases} \quad (38)$$

with separate factor  $j(\text{angles})$  depending on the angles of transformation but independent of the eigenvalues. The product structure of the Jacobian suggests the joint density function of eigenvalues to be in the form

$$P(E_1, \dots, E_N) = \frac{1}{Z} \left( \prod_{i < j}^N |E_i - E_j|^\beta \right) e^{-A \sum_i E_i^2} \quad (39)$$

with  $Z$  being a normalization factor.

### Wigner's semicircle law

The mean density of levels  $\nu(E)$  for any of the Gaussian ensembles equals, up to normalization, the marginal

single-level distribution

$$\nu(E) = \int dE_2 \dots dE_N P(E, E_2, \dots, E_N). \quad (40)$$

If the unit of energy is fixed such that

$$\langle |H_{ij}|^2 \rangle = \langle |H_{ii}|^2 \rangle = \frac{1}{4N} \quad (41)$$

the limit  $N \rightarrow \infty$  yields the semicircle law

$$\nu(E) = \begin{cases} \frac{2}{\pi} \sqrt{1 - E^2} & |E| < 1 \\ 0 & |E| > 1 \end{cases} \quad (42)$$

### Spacing Distributions

A popular spectral characteristic is the distribution of nearest-neighbor spacings  $S$  (here we assume  $S$  to be normalized to the average level spacing  $\Delta$ ) in a spectrum. For  $N = 2$  that distribution is easily obtained from the joint density of eigenvalues

$$P(S) = \frac{1}{Z} \int dE_1 dE_2 \delta(S - |E_1 - E_2|) |E_1 - E_2|^\beta e^{-A(E_1^2 + E_2^2)}. \quad (43)$$

Two parameters  $A, Z$  can be fixed by normalization  $\int_0^\infty dS P(S) = 1$  and choice of the energy scale  $\int_0^\infty dS S P(S) = 1$ .

**A) Gaussian Orthogonal Ensemble:** for this case  $\beta = 1$ . We first integrate over  $E_2$  by splitting this integral into two parts from  $(-\infty, E_1)$  and  $(E_1, \infty)$  and integrating with delta-function noticing that after integration remaining two integrals over  $E_1$  equal to each other we have:

$$P(S) = \frac{2}{Z} S e^{-\frac{AS^2}{2}} \int_{-\infty}^{\infty} dE_1 e^{-2A(E_1 - S/2)^2} = \quad (44)$$

$$= \frac{1}{Z} \sqrt{\frac{2\pi}{A}} S \exp\left(-\frac{AS^2}{2}\right).$$

Fixing constants  $A, Z$  by conditions mentioned above, we have  $A = \pi/2$  and  $Z = 2/\pi$  such that

$$P_{\text{GOE}}(S) = \frac{\pi S}{2} \exp\left(-\frac{\pi S^2}{4}\right) \quad (45)$$

**B) Gaussian Unitary Ensemble:** following the same procedure as above, assuming  $\beta = 2$  after splitting integral over  $E_2$  and integration with delta-function we have

$$P(S) = \frac{2}{Z} S^2 e^{-\frac{AS^2}{2}} \int_{-\infty}^{\infty} dE_1 e^{-2A(E_1 - S/2)^2} = \quad (46)$$

$$= \frac{1}{Z} \sqrt{\frac{2\pi}{A}} S^2 \exp\left(-\frac{AS^2}{2}\right).$$

Constants  $A, Z$  in this case turns out to be equal  $A = 8/\pi$  and  $Z = \pi^3/64$  which gives

$$P_{\text{GUE}}(S) = \frac{32S^2}{\pi^2} \exp\left(-\frac{4S^2}{\pi}\right) \quad (47)$$

- [1] M.L. Mehta, *Random Matrices*, 2nd edition (1991)
- [2] Joseph A. Zuk, arXiv:cond-mat/9412060
- [3] A.D. Mirlin, arXiv:cond-mat/0006421
- [4] Pier A. Mello *Theory of Random Matrices: Spectral Statistics and Scattering Problems*, Les Houches Lecture Session LXI "Mesoscopic Quantum Physics", (1994)