Phys 8501 Lecture 9 (T 10.05.2004)

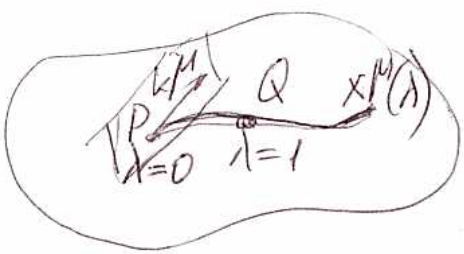
<u>Riemann normal coords</u>

Use geodesics to define locally inertial coords (i.e. $g_{\hat\mu\hat\nu}(p) = \eta_{\hat\mu\hat\nu}$, $\partial_\sigma g_{\hat\mu\hat\nu}|_p = 0$)

Define <u>exponential map</u> at p.

$$\exp_p : T_p \to M$$

$$k \to x^\mu(\lambda = 1), \text{ where } x^\mu(\lambda) \text{ solves geodesic equation}$$



$\underline{x^{\hat\mu}(Q) = k^{\hat\mu}}$, $x^{\hat\mu}(\lambda=1)|_Q = k^{\hat\mu}$

locally inertial coordinats – those that get mapped to the point Q.

$$\lambda k^{\hat\mu} = x^{\hat\mu}(\lambda)$$

tangent vector at p

$$\frac{d^2 x^{\hat\mu}}{d\lambda^2} = 0 = -\Gamma^{\hat\mu}_{\hat\rho\hat\sigma}(p)\left.\frac{dx^{\hat\rho}}{d\lambda}\right|_p \left.\frac{dx^{\hat\sigma}}{d\lambda}\right|_p$$

$$\Rightarrow \Gamma^{\hat\mu}_{\hat\rho\hat\sigma}(p) = 0$$

$$0 = \nabla_\sigma g_{\hat\mu\hat\nu} = \partial_\sigma g_{\hat\mu\hat\nu} - \cancelto{0}{\Gamma^{\hat\lambda}_{\hat\sigma\hat\mu} g_{\hat\lambda\hat\nu}} - \cancelto{0}{\Gamma^{\hat\lambda}_{\hat\sigma\hat\nu} g_{\hat\mu\hat\lambda}}$$

$\lambda k^{\hat\mu} = x^{\hat\mu}(\lambda) \leftarrow$ <u>Riemann normal coordinates</u>.

Example Expanding universe

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}\,dx^i dx^j$$

What are $\Gamma^\mu{}_{\nu\sigma}$?

$$I = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau$$

$$\delta I = \frac{1}{2}\int \delta\left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]d\tau$$

$$t \to t + \delta t \quad a(t+\delta t) = a(t) + \dot{a}\,\delta t$$

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\sigma g_{\mu\nu}\,\delta x^\sigma$$

$$\delta I = \frac{1}{2}\int\left[-2\frac{dt}{d\tau}\frac{d(\delta t)}{d\tau} + 2a\overbrace{\dot{a}\,\delta t}^{\delta a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]d\tau$$

$$= \int\left[\frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\delta t\,d\tau$$

$\delta I = 0$

$$\Rightarrow \quad \frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\mu = 0: \quad \frac{d^2t}{d\tau^2} + \Gamma^0_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\Rightarrow \Gamma^0_{00} = 0 = \Gamma^0_{i0} = \Gamma^0_{0i}$$

$$\Gamma^0_{ij} = a\mathring{a}\,\delta_{ij}$$

$\Gamma^i_{\rho\sigma}$ ~~look~~ at spatial variations

$$x^i \to x^i + \delta x^i$$

$$\delta I = \frac{a^2(t)}{2}\int \delta_{ij}\left[\frac{d\delta x^i}{d\tau}\frac{dx^j}{d\tau} + \frac{dx^i}{d\tau}\frac{d\delta x^j}{d\tau}\right]d\tau$$

integration by parts $= -\int\left(a^2\frac{d^2x^i}{d\tau^2} + 2a\frac{da}{d\tau}\frac{dx^i}{d\tau}\right)\delta_{ij}\,\delta x^j\,d\tau$

$$\frac{da}{d\tau} \to \frac{da}{dt}\frac{dt}{d\tau} = \mathring{a}\frac{dt}{d\tau}$$

$\delta I = 0 \Rightarrow$

$$\frac{d^2x^i}{d\tau^2} + \underbrace{\frac{2\mathring{a}}{a}}_{=\Gamma^i_{\rho\sigma}}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0$$

$$\Gamma^i_{00} = 0 = \Gamma^i_{jk}$$

$$\Gamma^i_{0j} = \frac{2\mathring{a}}{a}\delta^i_j = \Gamma^i_{j0}$$

$\Gamma^i_{0i} = \frac{2\mathring{a}}{a}$ <u>no summation</u>

null path $ds^2 = 0 \Rightarrow dt^2 = a^2 dx^2$ $y, z = \text{const}$

$$\Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}.$$

~~null~~ null geodesic $\frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx}{d\lambda}\right)^2 = 0$

$$\Rightarrow \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0 \Rightarrow \boxed{\frac{dt}{d\lambda} = \frac{\omega_0}{a}}$$

$E_{photon} = -p_\mu u^\mu \quad p^\mu = \frac{dx^\mu}{d\lambda}$

$= -g_{00}\frac{dt}{d\lambda}u^0 \quad u^\mu = (1,0,0,0)$

$\boxed{E = \frac{\omega_0}{a}}$ cosmological redshift.

— NOT a Doppler shift.

Energy-momentum tensor

$\nabla_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$

$\Rightarrow \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} = 0$

$\nu = 0: \quad \dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)$

$\nu = i: \quad \partial_i p = 0$

Assume equation of state $p = w\rho$

$$\frac{\dot\rho}{\rho} = -3\frac{\dot a}{a}(1+w) \Rightarrow \rho \propto a^{-3(1+w)}$$

matter $p=0$: $\rho \propto \frac{1}{a^3}$

radiation $p = \frac{1}{3}\rho$ $\quad \rho \propto \frac{1}{a^4}$

vacuum $p = -\rho \Rightarrow \rho \propto \text{const}$

$\quad$ ↳ cosmological constant.

Riemann Curvature tensor

$$\delta V^\rho = \underbrace{R^\rho{}_{\sigma\mu\nu}}_{(1,3)\text{ tensor - Riemann tensor}} V^\sigma A^\mu B^\nu$$

Note $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$

$$V^\rho \to V^\rho + \underline{\underline{\delta V^\rho}}$$

Use covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho$$

$$= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma -$$

$$-(\mu \leftrightarrow \nu) =$$

$$= \cancel{\partial_\mu \partial_\nu V^\rho} + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \cancel{\Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma} -$$

$$\nabla_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma) + \cancel{\Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - ($$

$$[\nabla_\mu, \nabla_\nu] V^\rho = \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) V^\sigma$$

$$- \underline{T^\lambda_{\mu\nu}} \nabla_\lambda V^\rho$$

torsion tensor
vanish if Γ - cristoffel symbol
(torsion free).

$$\equiv R^\rho{}_{\sigma\mu\nu} V^\sigma$$