Phys 8501 Lecture 3 (T 09.14.2004)

Tensors

A tensor τ of type (k,l) is a multilinear map

$$T : \underbrace{T_p^* \times \dots \times T_p^*}_{k} \times \underbrace{T_p \times \dots \times T_p}_{l} \to \mathbb{R}$$

$(0,1)$ tensor $\omega : T_p \to \mathbb{R}$ dual vector $\omega = \omega_\mu \hat{\theta}^{(\mu)}$
$\omega \in T_p^*$

$(1,0)$ tensor $V : T_p^* \to \mathbb{R}$ vector $V = V^\mu \vec{e}_{(\mu)}$
$V \in T_p^{**} \simeq T_p$

$V(\omega) = V^\mu \omega_\mu \in \mathbb{R}$
$\omega \in T_p^*$

Tensor product $\otimes$

T is (k,l) tensor, S is (m,n) tensor

$$T \otimes S\left(\omega^{(1)}, \dots, \omega^{(k)}, \dots, \omega^{(k+m)}, V^{(1)}, \dots, V^{(l+n)}\right)$$
$$= T\left(\omega^{(1)} \dots \omega^{(k)}, V^{(1)} \dots V^{(l)}\right) \times S\left(\omega^{(k+1)} \dots \omega^{(k+m)}, V^{(l+1)}, \dots, V^{(l+n)}\right)$$

$T \otimes S$ is a $(k+m, l+n)$ tensor

multiplication in $\mathbb{R}$.

3

In components $\quad T$ is (k,ℓ) tensor

$$T = \underbrace{T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}}_{\text{components}} \hat{e}_{(\mu_1)} \otimes \ldots \otimes \vec{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \ldots \otimes \hat{\theta}^{(\nu_\ell)}$$

k upper indices
ℓ lower indices

Under Lorentz transformation

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$$

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_\ell}{}_{\nu_\ell'} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$V : T_p^* \rightarrow \mathbb{R}$
$\omega : T_p \rightarrow \mathbb{R}$
$\omega(V) = \omega_\mu V^\mu \ldots$

Examples: (i) metric $\eta_{\mu\nu}$ is a $(0,2)$ tensor

i.e. $\eta(V,W) = \eta_{\mu\nu} V^\mu W^\nu \qquad V, W \in T_p$

$$= \underbrace{V \cdot W}_{\text{scalar} \in \mathbb{R}}$$

(ii) Kronecker delta δ_ν^μ of type $(1,1)$

inverse metric $\eta^{\mu\nu}$ $(2,0)$ tensor

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^{\mu}_{\rho} = \eta_{\rho\nu}\eta^{\nu\mu}$$

(iii) Levi-Civita symbol $(0,4)$ tensor

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ even permutation of } 0123 \\ -1 & \text{'' odd '' ''} \\ 0 & \text{otherwise} \end{cases}$$

Note
Only a tensor in Minkowski space!

Manipulating tensors

Contraction $(k,l) \to (k-1, l-1)$

$$\text{e.g. } S^{\mu\sigma}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu}$$

Can use metric η to raise & lower indices

$$\text{e.g. } T_{\mu\nu}{}^{\rho\sigma} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}T^{\alpha\beta}{}_{\gamma\delta}$$

$$T_{\mu\nu\alpha\beta} = \eta_{\alpha\rho}\eta_{\beta\sigma}T_{\mu\nu}{}^{\rho\sigma}$$

$$\eta_{\mu\nu} = \eta_{\nu\mu}$$

A tensor is symmetric in any of its indicies if ~~the~~ it is unchanged under exchange of those indices

A " " antisymmetric "

" " changes sign "

e.g $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ ~~symbols~~ symmetric in the 1st 2 indices

$T_{\mu\nu\rho} = -T_{\mu\rho\nu}$ antisymmetric in last two indices.

Symmetrize

eg $T_{(\mu_1 \cdots \mu_n)\rho}{}^{\sigma} = \frac{1}{n!}\Big[T_{\mu_1 \cdots \mu_n \rho}{}^{\sigma} +$ + sum over perms of $\mu_1 \ldots \mu_n \Big]$.

Antisymmetrization

$T_{[\mu_1 \cdots \mu_n]\rho}{}^{\sigma} = \frac{1}{n!}\Big[T_{\mu_1 \cdots \mu_n \rho}{}^{\sigma} +$ alternating sum over perms $\mu_1 \ldots \mu_n \Big]$

e.g. $T_{[\mu\nu]} = \frac{1}{2}\left[T_{\mu\nu} - T_{\nu\mu}\right]$

$S_{(\mu\rho)}{}^{\sigma} = \frac{1}{2}\left[S_{\mu\rho}{}^{\sigma} + S_{\rho\mu}{}^{\sigma}\right]$

Also

$T_{[\mu|\nu|\rho]} = \frac{1}{2}\left[T_{\mu\nu\rho} - T_{\rho\nu\mu}\right]$

antisymmetrize in μ and ρ:

For a (1,1) tensor $X^{\mu}{}_{\nu}$

the trace $X = X^{\lambda}{}_{\lambda}$

e.g. $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^{\mu}_{\mu} = 4$

(because $\eta^{\mu\nu}\eta_{\rho\nu} = \delta^{\mu}_{\rho}$)

$\eta^{\nu\mu}\eta_{\mu\nu} = \eta^{\mu}{}_{\mu}$ $\begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

Physics in Minkowski space

four-velocity

$u^{\mu} = \frac{dx^{\mu}}{d\tau}$ ← proper time four-momentum $p^\mu = m u^\mu$

energy $E = p^0$ and $p_\mu p^\mu = -m^2$

$E^2 = \vec{p}^2 + m^2$

$\eta = (-1 +1 +1 +1)$

force 4-vector $f^\mu = m \frac{dp^\mu}{d\tau}$

Define energy-momentum tensor

$T^{\mu\nu}$ = symmetric (2,0) tensor

physically "flux of 4-momentum p^μ accross a surface of const x^ν"

T^{00} – energy density ρ

$T^{0i} = T^{i0}$ = momentum density

T^{ij} ~~represent~~ viscosity

e.g. pressure $P_i = T^{ii}$

eg. <u>perfect fluid</u>

$$T^{\mu\nu} = \begin{pmatrix} \rho & & 0 \\ & p & \\ 0 & & p \; p \end{pmatrix} = (\rho + p) U^\mu U^\nu + p \eta^{\mu\nu}$$

p isotropic pressure

in cosmology

dust: $p = 0$, photons : $p = \frac{1}{3}\rho$, vac. energy: $p = -\rho$

$T^{\mu\nu}$ conserved $\boxed{\partial_\mu T^{\mu\nu} = 0}$

(continuity equation).

Read §1.10 Carroll