

Phys 8501 Lecture 2 (M 09.13.2004)

Spatial rotations and boosts

$$x^{M'} = \Lambda^{M'}_{\nu} x^{\nu}$$

or $X_{4 \times 1} = \Lambda_{4 \times 4} X_{4 \times 1}$ (matrix notation)

What are Λ ?

Require $(\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x')$

$\eta = \text{diag}(-1 +1 +1 +1) \leftarrow$ Lorentzian metric

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$(\Delta x)^T = (\Delta t, \Delta x, \Delta y, \Delta z)$$

$$\Rightarrow \boxed{\eta = \Lambda^T \eta \Lambda}$$

(*)

Matrices satisfying (*) are Lorentz transformations

Under matrix multiplication

have Lorentz group $\equiv O(3,1)$

Euclidean metric

c.f. $O(3)$ $\eta = \overbrace{(+1 +1 +1)}^3$

$\Lambda^T \Lambda = 1$

$SO(3)$ $\det \Lambda = 1$

(discrete parity transf. removed)

2

Note set of continuous Lorentz transformations
 $\det \Lambda = 1$ parity
 $\Lambda^0_0 \geq 1$ time reversed
Known as $SO(3,1)^\uparrow \equiv$ proper orthochronous
Lorentz group.

Examples: (i)

Rotation in x-y plane

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Boost in x-direction

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ see p14 Carroll

The set of translations and Lorentz transformations is ten-parameter non-Abelian group \equiv Poincaré group.

Vectors

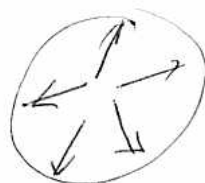
The tangent space T_p at each point p in spacetime is the set of all possible vectors located at that point!

$T_p =$ abstract vectorspace at each point in spacetime
 $\Rightarrow (a+b)(V+W) = aV + aW + bV + bW.$

This definition does not require an embedding of curved space in higher dimensions.



A vector field is a set of vectors with one at each spacetime point



Set of all tangent spaces is called tangent bundle

eg vector $A = A^M \underbrace{e_{(M)}}_{\text{basis}}$

Tangent vector $V = \frac{dx^M}{d\lambda} e_{(M)}$

where λ parametrises curve $x^M(\lambda)$



$$V = \frac{dx^\mu}{d\lambda} \vec{e}_\mu = \frac{dx^{\rho'}}{d\lambda} \vec{e}_{(\rho')} \Rightarrow$$

$$\Rightarrow \boxed{\vec{e}_{(\rho')} = \Lambda_{\rho'}^\mu \vec{e}_\mu} \quad \text{for } V \text{ to be invariant}$$

Dual vectors (One forms)

The dual vector space V^* is the set of all linear maps from the original vector space V to \mathbb{R}

$$\omega \in V^* \Rightarrow \omega: V \rightarrow \mathbb{R}$$

T_p^* = cotangent space

$$\text{if } \omega \in T_p^* \Rightarrow \omega: T_p \rightarrow \mathbb{R}$$

$$\omega(aV + bW) = a\omega(V) + b\omega(W)$$

where $a, b \in \mathbb{R}$, $V, W \in T_p$

Every dual vector ω can be written

$$\omega = \omega_\mu \overrightarrow{\hat{e}^\mu} \rightarrow \text{basis dual vector satisfying } \hat{e}^\mu(\vec{e}_\nu) = \delta^\mu_\nu$$

components with lower index label

(covariant vectors - old name)

2

Example: gradient of the scalar field

$$d\phi = \frac{\partial \phi}{\partial x^\mu} \partial^\mu \equiv \partial_\mu \phi \partial^\mu$$

$$d\phi V = \frac{\partial \phi}{\partial x^\mu} (\partial^\mu V^\nu \partial_\nu) =$$

$$= \frac{\partial \phi}{\partial x^\mu} V^\mu = \frac{\partial \phi}{\partial x^\mu} \frac{dx^\mu}{d\lambda} = \frac{d\phi}{d\lambda} \in \mathbb{R}$$

Tensors

A tensor of type (k, l) is a multilinear map from dual vectors and vectors to \mathbb{R}

$$T: \underbrace{T_p^* \times \dots \times T_p^*}_k \times \underbrace{T_p \times \dots \times T_p}_l \rightarrow \mathbb{R},$$

gen. vectors gen. dual vector

vectors $V: T_p^* \rightarrow \mathbb{R}$

$$V(\omega) = V^\mu \partial^\mu \omega = V^\mu \omega_\mu \in \mathbb{R}.$$

For $\omega \in T_p^*$

dual of the dual ^{vector} space gives the vector space.

$$T_p^{**} = T_p$$

vector space V

\mathcal{L}

dual space V^*

$$\omega \in V^* \quad \omega: V \rightarrow \mathbb{R}$$

dual (dual space) = V^{**}

$$\sigma \in V^{**} \quad \sigma: V^* \rightarrow \mathbb{R}$$

same as original
vector space V