Phys 8501 Lecture 2 (M 09.13.2004)

Spatial rotations and boosts

$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$

or $X_{4\times 1} = \Lambda_{4\times 4} X_{4\times 1}$ (matrix notation)

What are Λ ?

Require $(\Delta s)^2 = (\Delta x)^T \eta (\Delta x) = (\Delta x')^T \eta (\Delta x')$

$\eta = \mathrm{diag}(-1\ +1\ +1\ +1) \leftarrow$ Lorentzian metric

$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$(\Delta x)^T = (\Delta t, \Delta x\ \Delta y\ \Delta z)$

$\Rightarrow \boxed{\eta = \Lambda^T \eta \Lambda}$ $\quad (*)$

Matricies satisfying $(*)$ are Lorentz transformations

Under matrix multiplication have Lorentz group $\equiv O(3,1)$

c.f. $O(3)$ $\eta = \overbrace{(+1+1+1)}^{\text{Euclidean metric}}$

$\Downarrow$

$\Lambda^T \Lambda = 1$

$SO(3)$ $\det \Lambda = 1$

(descrete parity transf. removed)

Note set of continuous Lorentz transformations
det $\Lambda = 1$ parity
$\Lambda^{0'}{}_{0} \geq 1$ time reversed
Known as $SO(3,1)^{\uparrow}$ $\equiv$ proper orthochronous Lorentz group.

Examples: (i)
Rotation in x-y plane

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Boost in x-direction

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow$ see p 14 Carroll

The set of translations and Lorentz transformations is ten-parameter non-Abelian group $\equiv$ Poincaré group.

Vectors

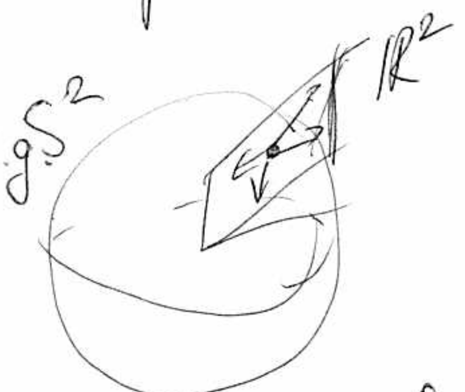
The tangent space T_p at each point p in spacetime is the set of all possible vectors located at that point!

T_p = abstract vectorspace at each point in spacetime

$$\Rightarrow (a+b)(V+W) = aV + aW + bV + bW.$$

This definition does not require an embedding of curved space in higher dimensions.

e.g. S^2



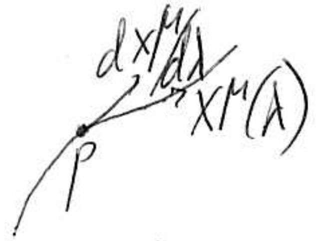
A vector field is a set of vectors with one at each spacetime point

Set of all tangent spaces is called tangent bundle

$$\text{eg vector } A = \underbrace{A^\mu}_{\text{components}} \underbrace{\hat{e}_{(\mu)}}_{\text{basis}}$$

$$\text{Tangent vector } V = \frac{dx^\mu}{d\lambda} \hat{e}_{(\mu)}$$

where λ parametrises curve $x^\mu(\lambda)$



$$V = \frac{dx^\mu}{d\lambda}\vec{e}_{(\mu)} = \frac{dx^{\nu'}}{d\lambda}\vec{e}_{(\nu')} \Rightarrow$$

$$\Rightarrow \boxed{\vec{e}_{(\nu')} = \Lambda^{\mu}_{\nu'}\,\vec{e}_{(\mu)}}$$ for V to be invariant

Dual vectors (One forms)

The dual vector space V^* is the set of all linear maps from the original vector space V to $\mathbb{R}$

$$\omega \in V^* \Rightarrow \omega : V \to \mathbb{R}$$

T_p^* = cotangent space

if $\omega \in T_p^* \Rightarrow \omega : T_p \to \mathbb{R}.$

$$\omega(aV + bW) = a\omega(V) + b\omega(W)$$

where $a, b \in \mathbb{R}$, $V, W \in T_p$

Every dual vector ω can be written

$$\omega = \underbrace{\omega_\mu}_{\text{components with lower index label}} \hat{\theta}^{(\mu)}$$

$\hat{\theta}^{(\mu)}$ → basis dual vector satisfying $\hat{\theta}^{(\nu)}\hat{e}_{(\mu)} = \delta^{\nu}_{\mu}$

(covariant vectors - old name)

<u>Example</u>: gradient of the scalar field

$$d\varphi = \frac{\partial \varphi}{\partial x^\mu} \hat{\theta}^{(\mu)} \equiv \partial_\mu \varphi \, \hat{\theta}^{(\mu)}$$

$$d\varphi\, V = \frac{\partial \varphi}{\partial x^\mu} \hat{\theta}^{(\mu)} V^\rho \hat{e}_{(\rho)} =$$

$$= \frac{\partial \varphi}{\partial x^\mu} V^\mu = \frac{\partial \varphi}{\partial x^\mu} \frac{dx^\mu}{d\lambda} = \frac{d\varphi}{d\lambda} \in \mathbb{R}$$

Tensors

A tensor of type (k, l) is a multilinear map from dual vectors and vectors to $\mathbb{R}$

$$T : \underbrace{\overset{\text{gen. vectors}}{T_p^* \times \ldots \times T_p^*}}_{k} \times \underbrace{\overset{\text{gen. dual vector}}{T_p \times \ldots \times T_p}}_{l} \to \mathbb{R}.$$

vectors $V : T_p^* \to \mathbb{R}$

$$V(\omega) = V^\mu \hat{e}_{(\mu)} \omega_\rho \hat{\theta}^{(\rho)} = V^\mu \omega_\mu \in \mathbb{R}.$$

For $\omega \in T_p^*$

dual of the dual vector space gives the vector space.

$$T_p^{**} = T_p$$

$\mathcal{L}$

vector space V

dual space V^*

$\omega \in V^*$ $\omega : V \to \mathbb{R}$

dual (dual space) $= V^{**}$:

$\sigma \in V^{**}$ $\underline{\sigma : V^* \to} \mathbb{R}$ = vector space V same as original