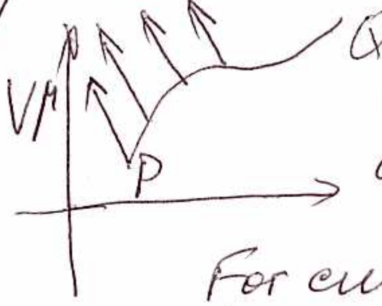
Phys 8501 Lecture 8 (M 09.04.2004)

Parallel Transport

Want notion of comparing tensors ~~of~~ at *different* spacetime points.

e.g. vectors in flat space



Cartesian components are kept const.

For curve $x^\mu(\lambda)$

$$\frac{dV^\mu}{d\lambda} = 0 = \frac{dx^\nu}{d\lambda}\,\partial_\nu V^\mu$$

partial derivatives are nontensorial

Thus, define *parallel transport* of a tensor T along some curve $x^\mu(\lambda)$ to be

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \equiv \underbrace{\frac{dx^\sigma}{d\lambda}}_{\text{tangent vector}}\ \underbrace{\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_k}}_{(k,\ell+1)\text{ tensor}} = 0.$$

$\partial_\sigma + \Gamma$

where $\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$ = directional covariant derivative

e.g. vector V^μ

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0.$$

Note (i): The metric is always parallel transported

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0.$$

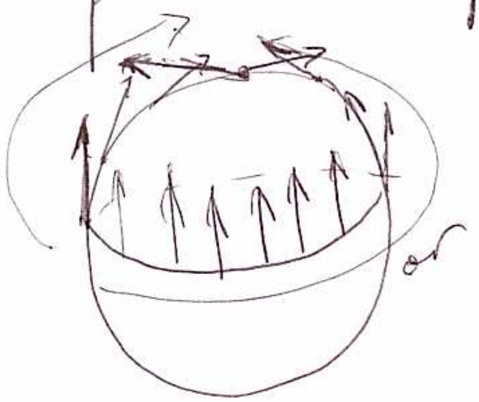
(ii) If V^μ and W^μ are parallel-transported along $x^\mu(\lambda)$ then:

$$\frac{D}{d\lambda}(g_{\mu\nu} V^\mu W^\nu) = 0$$

inner product is always parallel transported

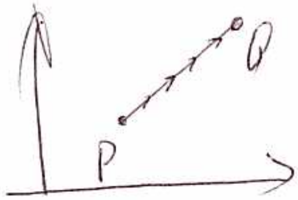
inner product is preserved along path.

parallel transport depends on a path:



Geodesics

In flat space: straight line = path of shortest distance between two points



or path that parallel transports its own tangent vector

A <u>geodesic</u> is a curve along which the <u>tangent vector</u> is parallel transported.

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda}=0$$

$$\Longrightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}=0}$$

When curvature is 0 $\Rightarrow$ straight line

<u>Note</u> Geodesic eqn is a 2nd order differential eqn
Need to specify $x^\mu(0)$ $\left.\frac{dx^\mu}{d\lambda}\right|_0$

<u>Alternative derivation</u>

Consider time-like path :

$$\tau = \int ds = \int\left(-g_{\mu\nu}dx^\mu dx^\nu\right)^{1/2} d\lambda$$

extremize:
$\delta\tau = 0$ variation with respect to

$$x^\mu \to x^\mu + \delta x^\mu$$
$$g_{\mu\nu} \to (\partial_\sigma g_{\mu\nu})\,\delta x^\sigma$$

$$\delta\tau = \int \frac{1}{2}\left(\quad\right)^{-1/2}\delta\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)d\lambda$$

we can choose the ~~representat~~ parametrisation of the path such that $(\quad)^{-1/2}$ ~~go~~ is 1.

$$\lambda = \tau \qquad \frac{dx^\mu}{d\tau} = u^\mu \qquad \underset{= g_{\mu\nu}u^\mu u^\nu}{u_\mu u^\mu} = -1$$

$$\delta I \equiv \frac{1}{2}\int \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)d\tau, \text{ where}$$

$$I = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau$$

$$\delta I = \frac{1}{2}\int\left[(\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d\,\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\,\delta x^\nu}{d\tau}\right]d\tau$$

$$\left(g_{\mu\nu}\to g_{\mu\nu}+(\partial_\sigma g_{\mu\nu})\delta x^\sigma\right)$$

After doing integration by parts for the 2nd and 3d term we get:

$$\delta I = -\int\left[g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu})\cdot\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]\delta x^\sigma d\tau$$

$$\delta I = 0 \Longrightarrow$$

$$\boxed{\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}=0}$$

here we get the Kristoffel ~~note~~ coefficients automatically

Note: Sometimes it is easier to calculate $\Gamma^\mu_{\rho\sigma}$ by $dI=0$ equation.

Properties of geodesics

The parametrisation along path defined up to

$\tau \to a\tau + b$ a, b consts

affine parameter

Free-falling particle with momentum $p^\mu = m u^\mu$ satisfies the geodesic equation:

$$p^\lambda \nabla_\lambda p^\mu = 0$$

Timelike geodesics are <u>maxima</u> of the proper time

(Recall timelike $ds^2 < 0 \Rightarrow d\tau^2 = -ds^2 > 0$)