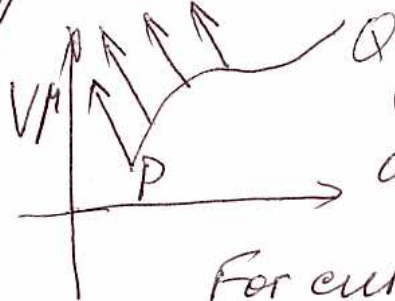


Parallel Transport

Want notion of comparing tensors ~~at~~ different spacetime points.

e.g. vectors in flat space



Cartesian components are kept const.

For curve $x^M(\lambda)$

$$\frac{dV^M}{d\lambda} = 0 = \frac{dx^D}{d\lambda} \partial_D V^M$$

Thus, define parallel transport of a tensor T along some curve $x^M(\lambda)$ to be

$$\left(\frac{D}{d\lambda} T \right)^{\mu_1 \dots \mu_k} \equiv \underbrace{\frac{dx^\delta}{d\lambda}}_{\text{tangent vector}} \underbrace{\nabla_\delta T^{\mu_1 \dots \mu_k}}_{(k+1)\text{ tensor}} = 0.$$

partial derivatives

where $\frac{D}{d\lambda} = \frac{dx^M}{d\lambda} \nabla_M =$ directional covariant derivative

e.g. vector V^M

$$\frac{dV^M}{d\lambda} + \Gamma^M_{\rho\sigma} \frac{dx^\sigma}{d\lambda} V^\rho = 0.$$

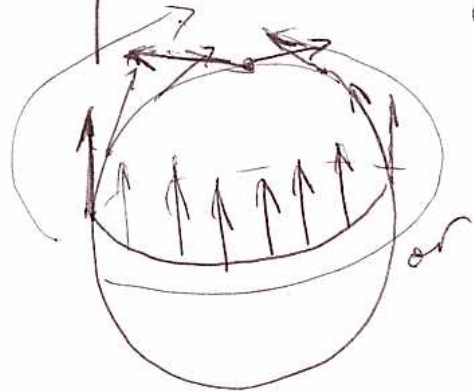
Note (i) : The metric is always parallel transported

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0.$$

(ii) If V^M and W^N are parallel-transported along $x^M(\lambda)$ then:

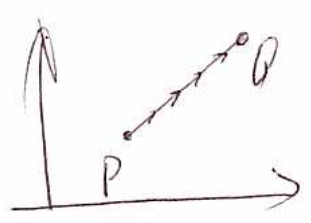
$$\frac{D}{d\lambda} (g_{\mu\nu} V^M W^N) = 0$$

inner product is always parallel transported
inner product is preserved along path.
parallel transport depends on a path:



Geodesics

In flat space : straight line = path of shortest distance between two points



or path that parallel transports its own tangent vector

A geodesic is a curve along which the tangent vector is parallel transported.

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = 0$$

$$\Rightarrow \boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0}$$

when curvature is 0 \Rightarrow straight line

Note Geodesic eqn is a 2nd order differential eqn
Need to specify $x^\mu(0)$ $\left. \frac{dx^\mu}{d\lambda} \right|_0$

Alternative derivation

Consider time-like path :

$x^\mu(\lambda)$

$$\tau = \int ds = \int (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2} d\lambda$$

extremize:

$$\delta \tau = 0 \quad \text{variation with respect to}$$

$$x^\mu \rightarrow x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \rightarrow (\partial_\alpha g_{\mu\nu}) \delta x^\alpha$$

$$\delta \tau = \int \frac{1}{2} ()^{-1/2} \delta (g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}) d\lambda$$

we can choose the ~~representat~~ parametrization of the path such that $()^{-1/2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ is 1.

$$\lambda = \tau$$

$$\frac{dx^\mu}{d\tau} = u^\mu$$

$$u_\mu u^\mu = -1$$

$$= g_{\mu\nu} u^\mu u^\nu$$

$$\delta I \equiv \frac{1}{2} \int \delta \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) d\tau, \text{ where}$$

$$I = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

$$\delta I = \frac{1}{2} \int \left[\delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d}{d\tau} \left(\frac{\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu \right) + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu \right] d\tau$$

$$(g_{\mu\nu} \rightarrow g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}) \delta x^\sigma)$$

After doing integration by parts for the 2nd and 3rd term we get:

$$\delta I = - \int \left[g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \cdot \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma d\tau$$

$$\delta I = 0 \Rightarrow$$

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0}$$

here we get the Christoffel ~~the~~ coefficients automatically

Note: Sometimes it's easier to calculate $\Gamma^\mu_{\nu\sigma}$ by $dI=0$ equation.

Properties of geodesics

The parametrisation along path defined up to

$$\mathbb{R} \rightarrow a\lambda + b \quad a, b \text{ const}$$

↑
affine parameter

Free-falling particle with momentum

$p^\mu = m u^\mu$ satisfies the geodesic equation:

$$p^\lambda \nabla_\lambda p^\mu = 0$$

Timelike geodesics are maxima of the proper time

(Recall timelike $ds^2 < 0 \Rightarrow \tau^2 = -ds^2 > 0$)