Phys 8501 Lecture 7 (T 09.28.2004)

Integration

$$d^n x = dx^0 \wedge \dots \wedge dx^{n-1} \leftarrow n\text{-form}$$

$$\Rightarrow \quad d^n x' = \underbrace{\left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|}_{\text{Jacobian}} d^n x$$

Invariant volume element

$$\sqrt{|g|'}\, d^n x' = \sqrt{|g|}\, d^n x$$

$$\varepsilon = \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_n}$$

$$= \frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$= \frac{1}{n!} \sqrt{|g|}\, \tilde{\varepsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$= \sqrt{|g|}\, dx^0 \wedge \dots \wedge dx^{n-1}$$

$$\Rightarrow \boxed{\varepsilon = \sqrt{|g|}\, d^n x}$$

Integral of scalar fn on n-dim manyfold

$$\boxed{\int_M \varphi(x)\sqrt{|g|}\,d^n x = \int_M \phi(x)\epsilon}$$

Covariant Derivative

want to generalize $\partial_\mu T^{\mu\nu}=0$.

Define covariant derivative operator ∇

want it to be independent of coordinates.

Require ∇ satisfy

(i) linearity $\nabla(T+S)=\nabla T+\nabla S$

(ii) Leibnitz rule

$$\nabla(T\otimes S)=(\nabla T)\otimes S+T\otimes(\nabla S).$$

$$\nabla_\mu V^\nu=\partial_\mu V^\nu+\underbrace{\Gamma^\nu_{\mu\lambda}}_{\text{connection coefficients}}V^\lambda$$

Demand that $\nabla_\mu V^\nu$ is a $(1,1)$ tensor

$$\nabla_{\mu'}V^{\nu'}=\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\nabla_\mu V^\nu$$

$$\Rightarrow \Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu}$$

$\Gamma^\nu_{\mu\lambda}$ are <u>NOT</u> components of a tensor

What about one forms?

Write $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda$

Require $\nabla_{\mu'} \omega_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \nabla_\mu \omega_\nu$

Impose two extra properties on ∇:

(iii) $\nabla_\mu \left(T^\lambda{}_{\lambda\rho}\right) = \left(\nabla T\right)_\mu{}^\lambda{}_{\lambda\rho}$

is ∇ commutes with contractions

(iv) $\nabla_\mu \varphi = \partial_\mu \varphi$.

Consider

$$\nabla_\mu \left(\omega_\lambda V^\lambda\right) = \left(\nabla_\mu \omega_\lambda\right) V^\lambda + \omega_\lambda \left(\nabla_\mu V^\lambda\right)$$

$$= \left(\partial_\mu \omega_\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda} \omega_\sigma\right) V^\lambda + \omega_\lambda \left(\partial_\mu V^\lambda + \Gamma^\lambda_{\mu\rho} V^\rho\right)$$

$$= \partial_\mu \left(\omega_\lambda V^\lambda\right) + \tilde{\Gamma}^\sigma_{\mu\lambda} \omega_\sigma V^\lambda + \Gamma^\lambda_{\mu\rho} V^\rho \omega_\lambda$$

$$\Rightarrow \tilde{\Gamma}_{\mu\lambda}^{\sigma}\, \omega_\sigma V^\lambda + \Gamma_{\mu\lambda}^{\sigma}\, \omega_\sigma V^\lambda = 0$$

$$\Rightarrow \boxed{\tilde{\Gamma}_{\mu\lambda}^{\sigma} = -\Gamma_{\mu\lambda}^{\sigma}}$$

For arbitrary tensor

$$\nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_e} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_e} +$$
$$+ \Gamma_{\sigma\lambda}^{\mu_1} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_e} + \Gamma_{\sigma\lambda}^{\mu_2} T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_e} + \ldots$$
$$- \Gamma_{\sigma\nu_1}^{\lambda} T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_e} - \Gamma_{\sigma\nu_2}^{\lambda} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_e} - \ldots$$

How to determine $\Gamma^{\mu}_{\sigma\lambda}$?

In 4 dimensions $4^3 = 64$ components.

Impose two additional properties

- <u>torsion-free</u> : $\Gamma_{\mu\nu}^{\lambda} = \Gamma_{(\mu\nu)}^{\lambda} \Rightarrow \Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}$

Define torsion tensor $T_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}$

- <u>metric-compatible</u> $\nabla_\rho g_{\mu\nu} = 0$.

$\nabla_\rho g^{\mu\nu} = 0$, $\nabla_\lambda \varepsilon_{\mu\lambda\rho\sigma} = 0$, $g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda} V^\lambda)$

$= \nabla_\rho V_\mu$

Now metric compatibility

$$\Rightarrow 0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} \quad (1)$$

$$0 = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} \quad (2)$$

$$0 = \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} \quad (3)$$

$$(1) - (2) - (3): \quad 0 = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho}$$

$$\Rightarrow \boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$

Unique connection derived from metric.

Known as Christoffel connection → use in GR!

(Riemannian connection)

Manifolds with metrics + connections →

Riemannian geometry

Example $ds^2 = dr^2 + r^2 d\theta^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{r\rho}\left(\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr}\right)$$

$$= \frac{1}{2} g^{rr}\left(2\partial_r g_{rr} - \partial_r g_{rr}\right) + \frac{1}{2} g^{r\theta}\left(2\partial_r g_{r\theta} - \partial_\theta g_{rr}\right.$$

$$\Gamma^r_{rr} = 0$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{r\rho}\left(\cancelto{0}{\partial_\theta g_{\theta\rho}} + \cancelto{0}{\partial_\theta g_{\rho\theta}} - \partial_\rho g_{\theta\theta}\right)$$

$$= -\frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -\frac{1}{2}(1)2r = -r$$

Divergence of a vector

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$$

$$= \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, V^\mu\right)$$

where $\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda |g|$

Obtain curved version of Stokes theorem:

$\Sigma \subset M$

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} x$$

Boundary

where n_μ is normal to $\partial\Sigma$

γ_{ij} is induced metric.