Phys 8501 Lecture 6 (M 09.27.04)

Locally inertial coords

Can always put an arbitrary metric $g_{\mu\nu}$ into canonical form $\mathrm{diag}(-1+++)$ at some point $p \in M$

i.e. $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$ and also $\underbrace{\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p)=0}_{\text{local Lorentz frame}}$

$x^{\hat{\mu}}$ = locally inertial coords

Deviation from flatness at 2nd order in derivatives of metric $\Longrightarrow$ curvature tensor

see p 74, 75 Carroll

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}$$

Tensor densities

Recall Levi-Civita symbol

$$\tilde{\varepsilon}_{\mu_1 \dots \mu_n} = \begin{cases} +1 & \text{even permut.} \\ -1 & \text{odd perm} \\ 0 & \text{otherwise} \end{cases}$$

Under ~~global~~ general coord. trans.

$$\tilde{\varepsilon}_{\mu_1' \dots \mu_n'} = \underbrace{\left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|}_{\text{determinant of matrix } \frac{\partial x^{\mu'}}{\partial x^{\mu}}} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \tilde{\varepsilon}_{\mu_1 \dots \mu_n}$$

Follows from

$$\tilde{\varepsilon}_{\mu_1' \cdots \mu_n'} |M| = \tilde{\varepsilon}_{\mu_1 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} \cdots M^{\mu_n}{}_{\mu_n'}$$

$\Longrightarrow \tilde{\varepsilon}$ is <u>not</u> a tensor

Known as <u>tensor density</u>

Similar, det. of metric $g_{\mu\nu}$ transforms as

$$g(x^{\mu'}) = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2} g(x^{\mu})$$

$\Longrightarrow |g_{\mu\nu}| \equiv g$ <u>is not</u> a tensor

$$\det g_{\mu'\nu'} \overset{\det}{=} \left[\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \right]$$

Convert tensor density into tensor
multiply by $|g|^n$ to appropriate power n.

<u>Define Levi-Civita tensor</u>

$$\varepsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\varepsilon}_{\mu_1 \cdots \mu_n}$$

Sim. $\varepsilon^{\mu_1 \cdots \mu_n} = \dfrac{1}{\sqrt{|g|}} \tilde{\varepsilon}^{\mu_1 \cdots \mu_n}$

$|g| = \| g_{\mu\nu} \|$ $\qquad \sqrt{-g}$

Differential forms

A *differential p-form* is an *antisymmetric* $(0,p)$ tensor

e.g. scalar = 0-form

dual vector ω_μ is 1-form

$\epsilon_{\mu\nu\rho\sigma}$ 4-form.

Wedge product

$$\underbrace{(A \wedge B)}_{p+q \text{ form}}{}_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

e.g. $A_{[\mu} B_{\nu]} = \frac{1}{2!}\left[A_\mu B_\nu - A_\nu B_\mu\right]$.

$(A \wedge B)_{\mu\nu} = 2 A_{[\mu} B_{\nu]}$

Note: $A \wedge B = (-1)^{pq} B \wedge A$

Exterior derivative

Note that $\partial_\mu W_\nu$ does not transfor as $(0,2)$

tensor

$$\partial_{\mu'} W_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu W_\nu + W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}$$

Define *exterior derivative* of p-form A

$$\boxed{(dA)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\,\partial_{[\mu_1} \underbrace{A_{\mu_2 \cdots \mu_{p+1}]}}_{p\text{-form}}}$$

d is a tensor in curved spacetime

e.g. $(dA)_{\mu_1\mu_2} = 2\,\partial_{[\mu_1}A_{\mu_2]} = \partial_{\mu_1}A_{\mu_2} - \partial_{\mu_2}A_{\mu_1}$

Note $d(dA) = 0$. or $d^2 = 0$.

A p-form A is closed if $dA = 0$ and exact if $A = dB$ for some $p-1$ form

Hodge duality

On n-dim manifold define Hodge $*$ operator for a p-form A:

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\,\epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p}$$

$\longrightarrow$ maps A into "A dual"

Note $**A = (-1)^{s+p(n-p)} A$.

where s = # of negative eigenvalues in metric

Similar to vectors and dualvectors except that Hodge dual depends on metric

$$(*(*A)_{\mu_1 \cdots \mu_{n-p}})_{\nu_1 \cdots \nu_p} = \epsilon^{\mu_1 \cdots \mu_{n-p}}{}_{\nu_1 \cdots \nu_p} (*A)_{\mu_1 \cdots \mu_{n-p}}$$

__Example__: Electromagnetism

Maxwell equations: $\partial_\mu F^{\mu\nu} = -J^\nu$

$F_{\mu\nu}$ = two-form $\quad \partial_{[\mu} F_{\nu\lambda]} = 0$

Equivalent to

exterior derivative

$$dF = 0$$
$$d(*F) = *J$$

F is exact $\Rightarrow F = dA$

where A is a 1-form = vector potential A_μ

Gauge invariance $A \to A + d\underbrace{\Lambda}_{0\text{-form}}$

$$F \to F$$

__Integration__

In $\mathbb{R}^n$ vol. element transforms

$$d^n x' = \underbrace{\left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right|}_{\text{Jacobian}} d^n x$$

On an n-dimensional manifold the integrand is an n-form

i.e. for $\Sigma \subset M$:

$$\int_\Sigma : \omega \to \mathbb{R}$$

In 1dim $\omega = \omega(x)dx$: $\int_\Sigma \omega(x)dx \in \mathbb{R}$

Thus we identify

$$d^n x = dx^0 \wedge \dots \wedge dx^{n-1}$$

$$= \frac{1}{n!} \tilde{\varepsilon}_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$= \frac{1}{n!} \tilde{\varepsilon}_{\mu_1 \dots \mu_n} \tilde{\varepsilon}_{\mu_1'} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} dx^{\mu_1'} \wedge \dots \wedge dx^{\mu_n'}$$

$$= \frac{1}{n!} \tilde{\varepsilon}_{\mu_1' \dots \mu_n'} \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| dx^{\mu_1'} \wedge \dots \wedge dx^{\mu_n'}$$

$$= \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| d^n x'$$

$$d^n x = \left| \frac{\partial x^\mu}{\partial x^{\mu'}} \right| d^n x'$$

The invariant volume element:

$$\sqrt{|g|}\, d^n x$$