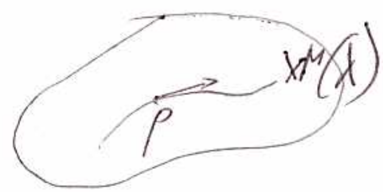
Vectors

Represent tangent vectors by "directional derivative ops"

Let $\mathcal{F} = (C^\infty \text{ maps } f: M \to \mathbb{R})$.

Each curve through $p \in M$ defines the <u>directional derivative operator</u>

$$\frac{d}{d\lambda} : \mathcal{F} \to \mathbb{R}$$
$$f \mapsto \left.\frac{df}{d\lambda}\right|_p$$



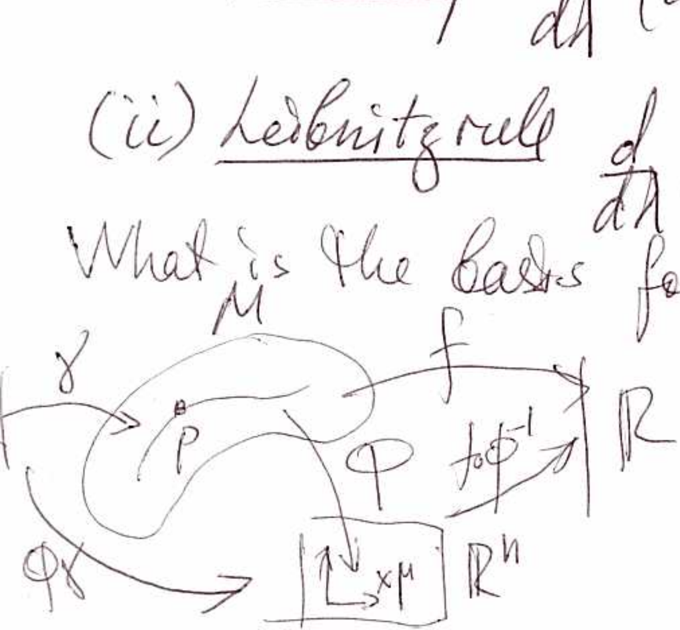
The <u>tangent space</u> T_p is space of directional derivative ops through p.

This follows because

(i) <u>Linearity</u> $\left.\frac{d}{d\lambda}(af+bg)\right|_p = a\left.\frac{df}{d\lambda}\right|_p + b\left.\frac{dg}{d\lambda}\right|_p$ $\quad a,b \in \mathbb{R}$, $f,g \in \mathcal{F}$

(ii) <u>Leibnitz rule</u> $\frac{d}{d\lambda}(fg) = f\left.\frac{dg}{d\lambda}\right|_p + g\left.\frac{df}{d\lambda}\right|_p$

What is the basis for T_p?



$$\frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \cdot \gamma) \qquad f \cdot \gamma : \mathbb{R} \to \mathbb{R}$$

$$= \frac{d}{d\lambda}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma\right) \qquad f \circ \varphi^{-1} : \mathbb{R}^4 \to \mathbb{R}, \quad \varphi \circ \gamma : \mathbb{R} \to \mathbb{R}^4$$

$$= \frac{d(\underbrace{\varphi \cdot \gamma}_{x^\mu})^\mu}{d\lambda} \cdot \frac{\partial f \circ \varphi^{-1}}{\partial x^\mu}$$

$$\frac{d}{d\lambda} f = \frac{dx^\mu}{d\lambda} \partial_\mu f$$

true for arbitrary fns f $\Longrightarrow$

$$\boxed{\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu}$$

Thus, $\frac{d}{d\lambda}$ is the tangent <u>vector</u> to the curve with parameter λ.

Under a change of co-ords

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

$$\frac{dx^{\mu'}}{d\lambda} = V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

In SR: $x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$

$$\frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu}$$

$$V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu}$$

For two vector fields

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Tensors

one form gradient of function $f = df$

$df \in T_p^*$

$df\left(\frac{d}{d\lambda}\right)$ $\quad df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}$

$\frac{d}{d\lambda} \in T_p$

Basis flat space satisfies $\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^{\mu}{}_{\nu}$

one-form basis

But $dx^{\mu}\partial_{\nu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu}$

$\Rightarrow$ $\{dx^{\mu}\}$ are a set of basis one-forms

They transform as: $dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}$

In general a tensor T is

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_\ell}$$

↑ инвариантный объект ↑ компоненты преобразуются

and

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \cdots \frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}} \times T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

<u>Metric tensor</u> "g"

new symbol $g_{\mu\nu}$ = symmetric $(0,2)$ tensor

$\det g_{\mu\nu} \equiv |g_{\mu\nu}| = g$

$g = |g_{\mu\nu}| \neq 0 \Rightarrow$ Inverse metric exists.

Inverse satisfies $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho = g_{\lambda\rho} g^{\lambda\mu}$

Expansion "g" $= g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$= g_{\mu\nu}\, dx^\mu dx^\nu \equiv$ "ds^2" line element

ds^2 is <u>not</u> the differential of anything.

So $V, W \in T_p$ $\quad g(V, W) = g_{\mu\nu} V^\mu W^\nu \equiv ds^2(V, W)$

Similarily $(dx)^2$ referes to $(0,2)$ tensor
$dx \otimes dx$

Example: Expanding universe

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$

($a^2(t)$: scale factor)

(exponential scale factor → inflation)

Typically: $a(t) = t^q$ (matter dominated $q = \frac{2}{3}$, radiation dominated $q = \frac{1}{2}$)

Light cones obtained from $ds^2 = 0$

$$\Rightarrow 0 = -dt^2 + t^{2q} dx^2 \quad \text{(assume } y, z \text{ const)}$$

$$\Rightarrow \frac{dx}{dt} = \pm t^{-q}$$

But strictly

$$dt^2(V,V) = (dt \otimes dt)(V,V) \quad V \in T_p$$
$$= dt(V)\, dt(V)$$
$$= \left(\frac{dt}{d\lambda}\right)^2$$

$$V = \frac{dx^\mu}{d\lambda} \partial_\mu$$

$$dt(V) = dt\left(\frac{dx^\mu}{d\lambda} \partial_\mu\right)$$
$$= \frac{dx^\mu}{d\lambda} dt(\partial_\mu)$$

$$= \frac{dx^\mu}{d\lambda}\frac{\partial t}{\partial x^\mu} = \frac{dt}{d\lambda}.$$

Similarily, $(dx^2)(V,V) = \left(\frac{dx}{d\lambda}\right)^2$

$$\Rightarrow 0(V,V) = 0 = -\left(\frac{dt}{d\lambda}\right)^2 + t^{2q}\left(\frac{dx}{d\lambda}\right)^2$$

$$\Rightarrow \left(\frac{dt}{d\lambda}\right)\left(\frac{dx}{d\lambda}\right)^{-1} = \frac{dx}{dt} = \pm t^{-q}$$

Lesson Expressions describe well-defined tensor relation, but manipulating one forms as "differentials" does give right answer (most of the time)

$$t = (1-q)^{1/1-q}\,(\pm x - x_0)^{\frac{1}{1-q}}$$