Phys 8501 Lecture 19 (T 11.09.2004)

Stars and Black holes

Interior metric of massive object requires $T_{\mu\nu} \neq 0$

Assume star is a perfect fluid

$$T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p\, g_{\mu\nu}$$

Look for static, spherically symmetric solution

ansatz $ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$

$\rightarrow$ must solve $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}\, 8\pi G$

Use $U_\mu = (e^\alpha, 0, 0, 0)$

$(U^\mu U_\mu = -1)$

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & 0 \\ & e^{2\beta} p & & \\ & & r^2 p & \\ 0 & & & r^2 \sin^2\theta\, p \end{pmatrix}$$

Einstein eq^{n}s

$$tt : \frac{1}{r^2} e^{-2\beta} \left(2r\partial_r\beta - 1 + e^{2\beta} \right) = 8\pi G \rho$$

$$rr : \frac{1}{r^2} e^{-2\beta} \left(2r\partial_r\alpha + 1 - e^{2\beta} \right) = 8\pi G p$$

$\theta\theta: \; e^{-2\beta}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right) = 8\pi G p$

$\phi\phi \propto \theta\theta$

Write tt equ as $\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$.

where $m(r) = \frac{r}{2G}\left(1 - e^{-2\beta(r)}\right)$.

Thus $m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$

For star of radius R and mass M

$$M = m(R) = \int_0^R \rho(r) \underbrace{4\pi r^2 dr}_{d^3\vec{x}}$$

But metric is curved not flat

$\gamma_{ij} dx^i dx^j = e^{2\beta} dr^2 + r^2 d\Omega^2$

$d^3x \to \sqrt{\gamma}\, d^3x = e^{\beta} r^2 \sin\theta\, d\theta\, d\phi$

$$\Rightarrow \bar{M} = \int_0^R \rho(r) e^{\beta(r)} 4\pi r^2 dr$$

$$\bar{M} = 4\pi \int_0^R \rho(r) \frac{r^2 dr}{\sqrt{1 - \frac{2Gm(r)}{r}}}$$

where $E_B = \bar{M} - M$ is the binding energy rr equ $$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi r^2 G p}{r(r-2Gm(r))}$$

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow (\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}$$

$$\Rightarrow \boxed{\frac{dp}{dr} = -(\rho+p)\frac{\left[Gm(r) + 4\pi G r^3 p\right]}{r(r-2Gm(r))}}$$

Tolman-Oppenheimer-Volkoff equ.

$$\boxed{\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)}$$

Normally $p = K\rho^\gamma$

K, γ - const.

<u>Example</u> Assume incompressible fluid

$\Rightarrow$ density is constant $\equiv \rho_*$

i.e. $$\rho(r) = \begin{cases} \rho_* & r<R \\ 0 & r>R \end{cases}$$

$$\Rightarrow m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_* & r<R \\ \frac{4}{3}\pi R^3 \rho_* = M & r>R \end{cases}$$

Integrating TOV eqn

$\underline{r<R}$ $p(r) = \rho_* \left[\frac{R\sqrt{R(R-2GM)} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}} \right]$

The interior metric $(r<R)$

$$ds^2 = -\left[\frac{3}{2}\sqrt{1-\frac{2GM}{r}} - \frac{1}{2}\sqrt{1-\frac{2GMr^2}{R^3}}\right]^2 dt^2 + \underbrace{\left(1-\frac{8\pi}{3}G\rho_* r^2\right)^{-1}}_{\sim\left(1-\frac{2Gm(r)}{r}\right)^{-1}} dr^2 + r^2 d\Omega^2$$

Note that pressure increases near core of star

$$p(0) = \rho_* \frac{\sqrt{R-2GM} - \sqrt{R}}{\sqrt{R} - 3\sqrt{R-2GM}}$$

Max. pressure occures when

$$\sqrt{R} = 3\sqrt{R-2GM}$$

$$\Rightarrow \boxed{M_{max} = \frac{4R}{9G}}$$

If $M > M_{max} \Rightarrow$ no static solution

$\Rightarrow$ star shrinks.

Buchdahl's theorem

Buchdahl's theorem = any reasonable static solution (spherically symmetric interior) has

$$M < M_{max} = \frac{4R}{9G}$$

How can black holes form?

Massive stars are ~~sp~~ supported by pressure from fusion burning.

After burning stops collapse halted by electron degeneracy pressure

$\longrightarrow$ known as white dwarf (typical size ~ Earth)

If $M > 1.44\, M_\odot$ = Chandrasekhar limit (1931)

then form neutron star (typical size ~ 10km)

Gravity force $p + e^- \to n + \nu_e$

$\longrightarrow$ rapidly spinning known as pulsars (1967)

If $M > 3\text{-}4\, M_\odot$ = Oppenheimer-Volkoff limit

then black hole

How can the black holes be seen?