Phys 8501 Lecture 18 (M 11.08.2004)

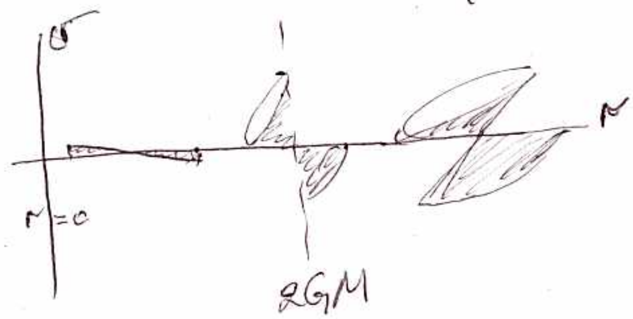
Eddington - Finkelstein coords:

$u = t - r^*$ $\qquad r \to 2GM \Rightarrow r^* \to -\infty$

$\qquad\qquad\qquad\qquad \Rightarrow t \to -\infty$

$v = t + r^*$

(v,r) $\quad ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dr\,dv + dv\,dr) + r^2 d\Omega^2$



Consider (u, r, θ, φ) coords

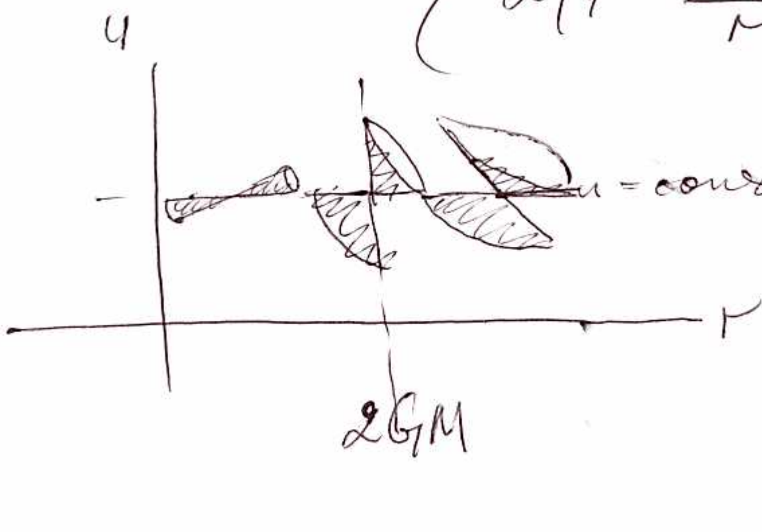
$$ds^2 = -\left(1 - \frac{2GM}{r}\right)du^2 - (du\,dr + dr\,du) + r^2 d\Omega^2$$

θ, φ const

radial null curves

$$ds^2 = 0$$

$$\Rightarrow \frac{du}{dr} = \begin{cases} 0 & \text{outgoing} \\ -2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{ingoing} \end{cases}$$



white hole

A set of coords that covers entire Schwarzschild manifold are the <u>Kruskal - Szekeres</u> (1960) coordinates (T, R, θ, φ)

$$\boxed{ds^2 = \frac{32G^2M^3}{r} e^{-r/2GM}\left(-dT^2+dR^2\right)+r^2 d\Omega}$$

$$T = \left(\frac{r}{2GM}-1\right)^{1/2} e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right)$$

$$R = \text{---}\ \text{''}\ \text{---}\ \text{''}\ \text{---} \cosh\left(\quad\right)$$

<u>Properties</u>

① radial null curves satisfy

$$\frac{dT}{dR} = \pm 1$$

$$\Rightarrow T = \pm R$$

② $T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$

event horizon $r = 2GM$

located at $T^2 - R^2 = 0$

$$\Rightarrow T = \pm R$$

③ $r = \text{const} \Rightarrow T^2 - R^2 = \text{const}$ a hyperbolae

$t = \text{const} \Rightarrow \frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) = \text{const}$

straight lines though origin

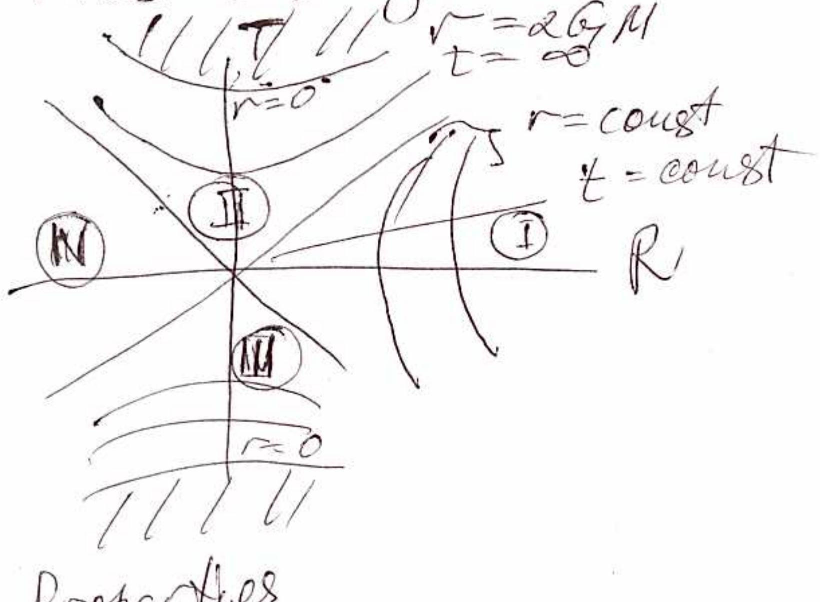
At $t \to \pm\infty$ $\quad \frac{T}{R} = \pm 1$ = event horizon

(4) Range of coords:

$$-\infty < R < \infty$$

$$T^2 < 1 + R^2 \Rightarrow |T| < \sqrt{1+R^2}$$

Kruskal diagram



Properties

(I) Original Schw. coords are good in this region i.e. $r > 2GM$

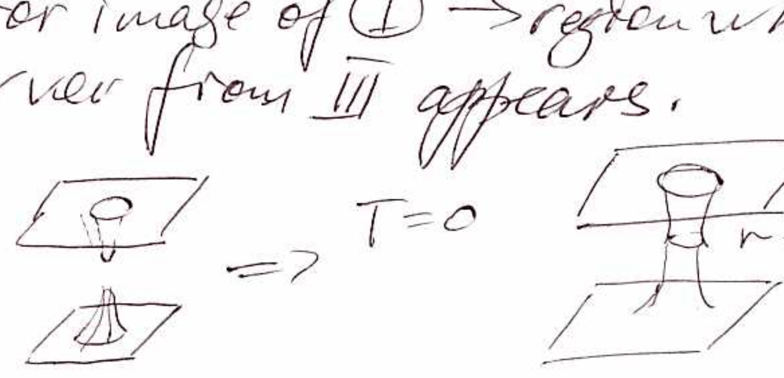
(II) Black hole region ,

Observer crossing null ~~line~~ $R = T$ never returns

(III) Has exactly time-reversed properties of II → white hole

(IV) Mirror image of (I) → region where ~~the~~ observer from III appears.

Const T slices



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Penrose diagrams (Appendix H)

Conformal transformation $g_{\mu\nu} \rightarrow \omega^2(x) g_{\mu\nu}$

These diagrams have standard properties

① There is a "timelike" and "spacelike" coord

② radial null rays to satisfy $\frac{dT}{dR} = \pm 1$

null condition $0 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \omega^2 g'_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$

③ Infinity is mapped to a finite point.

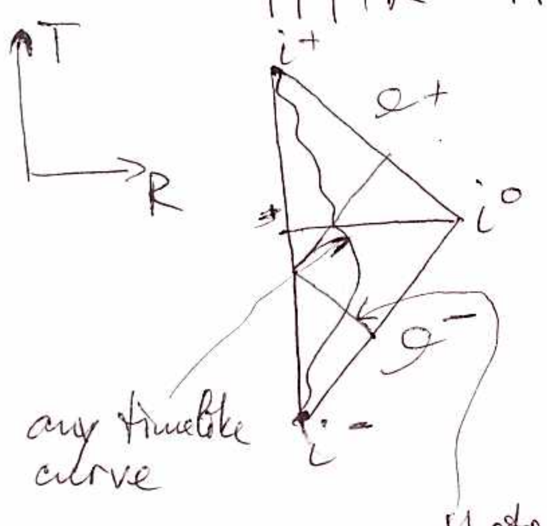
e.g. Minkowski space

$$ds^2 = -dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{r^2 d\Omega}$$

change coords

$$d\bar{s}^2 = \omega^2(T,R)\, ds^2$$
$$= -dT^2 + dR^2 + \sin^2 R\, d\Omega$$

$$0 \leq R < \pi$$
$$|T| + R < \pi$$



Conformal infinity has following regions

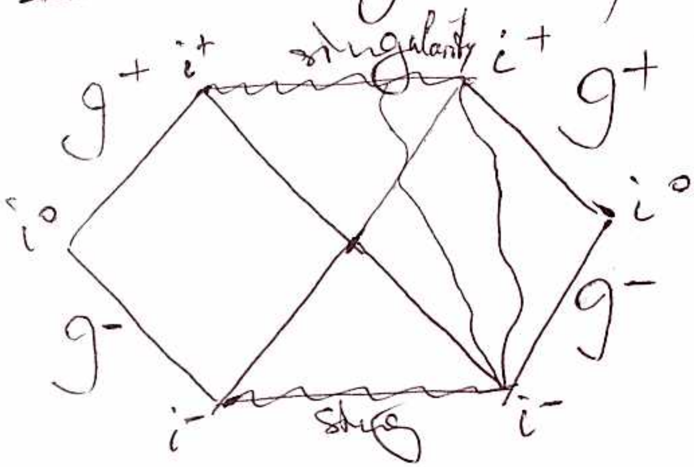
i^+ – future timelike infinity

i^0 – future spacelike infinity

i^- – past timelike infinity

script $\mathscr{I}$ $\begin{cases} \mathscr{I}^+ = \text{future null inf} \\ \mathscr{I}^- = \text{past null inf.} \end{cases}$

Penrose diagram for Schw. spacetime



$$(u', v', \theta, \varphi) \to (u'', v'', \theta, \varphi)$$

Thursday
Friday } $5^{30}, 6^{00}$?