Phys 8501 Lecture 16 (M 11.01.2004)

$$ds^2 = -\left(1-2\frac{GM}{r}\right)dt^2 + \left(1-2\frac{GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$\ell_p = 10^{-33}$ cm $\hbar, c, G$

quantum gravity effects important

$$\left.\begin{aligned} mc^2 = E = \frac{\hbar c}{\lambda} \Rightarrow \lambda = \frac{\hbar}{mc} \\ GR \longrightarrow r = \frac{GM}{c^2} \end{aligned}\right\} \Rightarrow \overset{\lambda \simeq r}{\frac{\hbar}{mc} \simeq \frac{GM}{c^2}}$$

10^{19} GeV 10^{-33} cm

$S = \int \sqrt{-g} \ldots$ $R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$

$$S = \int d^4x \sqrt{-g}\, M_p^2 \left[R + \frac{1}{M_p^2} R_{\mu\nu} R^{\mu\nu} + \ldots \right]$$

$\ell \gtrsim \ell_p$

$E \lesssim E_p$

$$S = \int d^4x \sqrt{-g} \left[\Lambda + M_p^2 R + R_{\mu\nu}R^{\mu\nu} + \ldots \right.$$

$\sqrt{R_{\mu\nu}R^{\mu\nu}}$ locality

<u>Geodesics</u> $$\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

$x^\mu = (t, r, \theta, \varphi)$

Instead use symmetries of solution

$\Rightarrow$

$\Rightarrow$ 4 Killing vectors = 1 time translation
+ 3 rotation

Time translations $\rightarrow$ conservation of energy, E

$K^\mu = (\partial_t)^\mu = (1,0,0,0)$

$E = -K_\mu \frac{dx^\mu}{d\lambda} = -g_{\mu\nu} K^\mu \frac{dx^\nu}{d\lambda}$

$$\boxed{E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}}$$

Spatial rotations $\rightarrow$ conservation of angular momentum L.

$\rightarrow$ particle moves in a plane (choose $\theta = \frac{\pi}{2}$)
$\rightarrow$ the magnitude of the angular momentum has Killing vector

$R^\mu = (\partial_\varphi)^\mu = (0,0,0,1)$

$L = R_\mu \frac{dx^\mu}{d\lambda} = g_{\mu\nu} R^\mu \frac{dx^\nu}{d\lambda}$
$= r^2 \sin^2\theta \frac{d\varphi}{d\lambda}$

$$\boxed{L \overset{\theta = \frac{\pi}{2}}{=} r^2 \frac{d\varphi}{d\lambda}}$$

Expanding inner product

$$-\varepsilon = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \qquad \varepsilon = \begin{cases} 0 & \text{null} \\ 1 & \text{timelike} \end{cases}$$

$$= -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$\Rightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right) = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \leftarrow \text{Goldstein p. 79.}$$

where $\mathcal{E} = \frac{1}{2}E^2$

$$V(r) = \underbrace{\frac{1}{2}\varepsilon - \varepsilon\frac{GM}{r} + \overbrace{\frac{L^2}{2r^2}}^{\text{centrifugal barrier}}}_{\text{Newtonian}} \underbrace{- \frac{GML^2}{r^3}}_{GR}$$

Circular orbits $\Rightarrow \frac{dV}{dr} = 0$

$$\varepsilon GM r_c^2 - L^2 r_c + 3GML^2\gamma = 0$$

$$\gamma = \begin{cases} 0 & \text{Newtonian} \\ 1 & GR \end{cases}$$

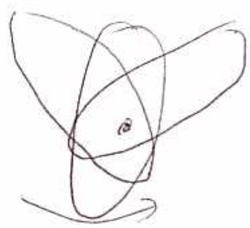
	$\gamma = 0$	$\gamma = 1$ (GR)
Massless particles ($\varepsilon = 0$)	no circular orbits	$3GM$ (unstable)
Massive particles	$r_c = \frac{L^2}{GM}$	$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}$

$r > 6GM$ stable

$3GM < r < 6GM$ unstable

Experimental tests

1. Precession of perihelia



$r = r(\varphi)$

Use eqn for $\frac{dr}{d\varphi}$

$$L^2 = r^4 \left(\frac{d\varphi}{d\lambda}\right)^2$$

Obtain

$$\left(\frac{dr}{d\varphi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM r^3}{L^2} + r^2 - 2GMr = 2E\frac{r^4}{L^2}$$

Define $\quad x = \frac{L^2}{GMr}$

$$\left(\frac{dx}{d\varphi}\right)^2 + \frac{L^2}{G^2M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \frac{2EL^2}{G^2M^2}$$

Expand $\quad x = x_0 + x_1$

$$x_0 = 1 + e\cos\varphi$$

e ← eccentricity

$$\Rightarrow \frac{d^2x_1}{d\varphi^2} + x_1 = 3\frac{G^2M^2}{L^2}x_0^2$$

$$\Rightarrow x = 1 + e\cos\varphi + 3\frac{G^2M^2}{L^2}e\varphi\sin\varphi$$

$$= 1 + e\cos[(1-\alpha)\phi]$$

$$\boxed{\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2} \simeq \frac{6\pi G^2 M^2}{c^2(1-e)a}}$$

where $L^2 \simeq GM(1-e^2)a$

<u>Mercury</u> : $\frac{GM_\odot}{c^2} = 1.48 \times 10^5$ cm

orbital parameters $\begin{cases} a = 5.79 \times 10^{12} \text{ cm} \\ e = 0.2056 \end{cases}$

$\Rightarrow \Delta\phi'' = 0.103''$/orbit

or $43''$/century.

Agrees with the measurement !!

dark matter LHC

MOND

Modified Newtonian dynamics $F = ma \rightarrow \frac{ma^2}{a_0}$