

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$l_p = 10^{-33} \text{ cm } \quad \hbar, c, G$$

quantum gravity effects important

$$m c \lambda = E = \frac{\hbar c}{\lambda} \Rightarrow \lambda = \frac{\hbar}{m c}$$

$$GR \rightarrow r = \frac{GM}{c^2}$$

$$\Rightarrow \lambda \approx r \Rightarrow \frac{\hbar}{m c} \approx \frac{GM}{c^2}$$

$$10^{19} \text{ GeV} \quad 10^{-33} \text{ cm}$$

$$S = \int \sqrt{-g} \dots$$

$$S = \int d^4x \sqrt{-g} M_p^2 \left[R + \frac{1}{M_p^2} R_{\mu\nu} R^{\mu\nu} + \dots \right]$$

$$l \geq l_p$$

$$E \lesssim E_p$$

$$S = \int d^4x \sqrt{-g} \left[\Lambda + M_p^2 R + \sqrt{R_{\mu\nu} R^{\mu\nu}} + \dots \right] \quad \text{locality}$$

Geodesics $\frac{dx^\mu}{d\lambda} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

$$x^M = (t, r, \theta, \phi)$$

Instead use symmetries of solution
 \Rightarrow

\Rightarrow 4 Killing vectors = 1 time translation
+ 3 rotation

Time translations \rightarrow conservation of energy, E

$$K^M = (\partial_t)^M = (1, 0, 0, 0)$$

$$E = -K_M \frac{dx^M}{dt} = -g_{\mu\nu} K^M \frac{dx^\nu}{dt}$$

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{dt}$$

Spatial rotations \rightarrow conservation of angular momentum L .

- \rightarrow particle moves in a plane (Choose $\theta = \frac{\pi}{2}$)
- \rightarrow the magnitude of the angular momentum has Killing vector

$$R^M = (\partial_\phi)^M = (0, 0, 0, 1)$$

$$L = R_M \frac{dx^M}{dt} = g_{\mu\nu} R^M \frac{dx^\nu}{dt}$$

$$= r^2 \sin^2 \theta \frac{d\phi}{dt}$$

$$L \Big|_{\theta = \frac{\pi}{2}} = r^2 \frac{d\phi}{dt}$$

Expanding inner product

$$-\epsilon = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \quad \epsilon = \begin{cases} 0 & \text{null} \\ < 1 & \text{timelike} \end{cases}$$

$$= -\left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right) \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\phi}{d\lambda}\right)^2$$

$$\Rightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right) \left(\frac{L^2}{r^2} + \epsilon\right) = 0$$

$$\Rightarrow \frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \quad \leftarrow \text{Goldstein p. 79.}$$

where $\mathcal{E} = \frac{1}{2} E^2$

$$V(r) = \underbrace{\frac{1}{2} \mathcal{E} - \mathcal{E} \frac{GM}{r}}_{\text{Newtonian}} + \underbrace{\frac{L^2}{2r^2}}_{\text{centrifugal barrier}} - \underbrace{\frac{GM L^2}{r^3}}_{\text{GR}}$$

Circular orbits $\Rightarrow \frac{dV}{dr} = 0$

$$\mathcal{E} GM r_c^2 - \frac{L^2}{r_c} + 3GM L^2 \gamma = 0$$

$\gamma = 0$ Newtonian
 < 1 GR

$\gamma = 0$

$\gamma = 1$ (GR)

Massless particles
 $(\epsilon = 0)$

no circular orbits

$3GM$ (unstable)

Massive particles

$$r_c = \frac{L^2}{GM}$$

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12GM^2 L^2}}{2GM}$$

$r > 6GM$ stable

$3GM < r < 6GM$ unstable

Experimental Tests

1. Precession of perihelia



Use eqn for $\frac{dr}{d\phi}$

$$L^2 = r^4 \left(\frac{d\phi}{dr}\right)^2$$

Obtain

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2} r^3 + r^2 - 2GM/r = \frac{2EM^2}{L^2}$$

Define

$$x = \frac{L^2}{GM r}$$

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2 M^2} - 2x + x^2 - \frac{2GM^2}{L^2} x^3 = \frac{2EL^2}{G^2 M^2}$$

Expand

$$x = x_0 + x_1$$

$$x_0 = 1 + e \cos \phi$$

↑ eccentricity

$$\Rightarrow \frac{d^2 x_1}{d\phi^2} + x_1 = 3 \frac{GM^2}{L^2} x_0^2$$

$$\Rightarrow x = 1 + e \cos \phi + 3 \frac{GM^2}{L^2} e \phi \sin \phi$$

$$= 1 + e \cos[(1 - \alpha)\phi]$$

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G^2 M^2}{L^2} \approx \frac{6\pi G^2 M^2}{c^2(1-e)a}$$

where $L^2 \approx GM(1-e^2)a$

Mercury

$$\frac{GM_{\odot}}{c^2} = 1.48 \times 10^5 \text{ cm}$$

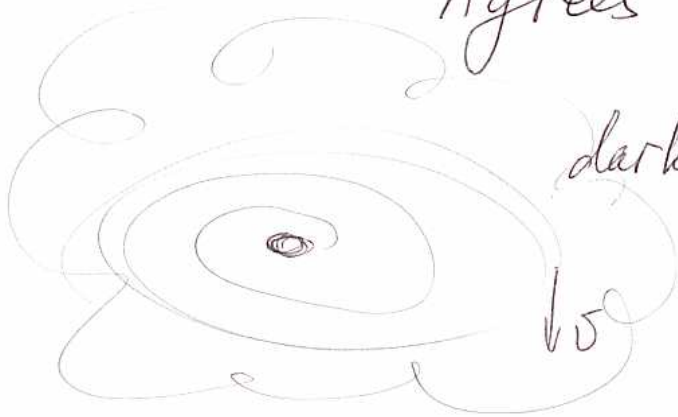
orbital parameters

$$\left\{ \begin{array}{l} a = 5.79 \times 10^{12} \text{ cm} \\ e = 0.2056 \end{array} \right.$$

$$\Rightarrow \Delta\phi'' = 0.103''/\text{orbit}$$

or 43''/century.

Agrees with measurement!!



dark matter LHC

↓

MOND

Modified Newtonian dynamics $F = ma \rightarrow \frac{ma^2}{a_0}$