

The Schwarzschild Solution

Einstein's ^{equation} in a vacuum:



$$R_{\mu\nu} = 0$$

$$\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \right) \quad \text{You can take Trace and show } R = 0$$

Look for solutions of $g_{\mu\nu}$ that are static and spherically symmetric.

static $\Rightarrow g_{\mu\nu}$ independent of t .
(and no cross terms; $dx dt \rightarrow$ assume symmetry $t \rightarrow -dt$)

spherically symmetric

\Rightarrow use polar coordinates

$$(t, r, \theta, \phi)$$

Start with ansatz

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Define $r' = e^{\gamma(r)} r$

$$dr' = e^{\gamma} dr + r e^{\gamma} \frac{d\gamma}{dr} dr = e^{\gamma} \left(1 + r \frac{d\gamma}{dr} \right) dr$$

$$\Rightarrow ds^2 = e^{2\alpha(r)} dt^2 + \underbrace{\left(1 + r \frac{d\gamma}{dr} \right)^2 e^{2\beta(r)} r^2}_{e^{2\beta'(r)}} dr'^2 + r'^2 d\Omega^2$$

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$$\Rightarrow ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

Need to solve for $\alpha(r)$ and $\beta(r)$

Substituting $g_{\mu\nu}$ into $\Gamma^{\mu}_{\rho\sigma}$ and $R^{\rho}_{\mu\nu\lambda}$, obtain

$$0 = R_{tt} = e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right]$$

$$0 = R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$0 = R_{\theta\theta} = e^{2\beta} \left[r(\partial_r \beta - \partial_r \alpha) - 1 \right] + 1$$

$$0 = R_{\phi\phi} = \sin^2 \theta R_{\theta\theta}$$

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr}$$

$$= \frac{2}{r} \partial_r (\alpha + \beta) \Rightarrow \alpha + \beta = \text{const} = c$$

$$ds^2 = -e^{-2\beta} e^{2c} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \dots$$

$$t' = e^c t$$

Absorb c by rescaling $t \rightarrow e^{-c} t$

$$\alpha = -\beta$$

Next $R_{\theta\theta} = 0$

$$\Rightarrow e^{2\alpha} [r(-2)\partial_r \alpha - 1] + 1 = 0$$

$$\frac{d}{dr} (r e^{2\alpha}) = 1$$

$$\Rightarrow r e^{2\alpha} = r + C$$

$$\Rightarrow e^{2\alpha} = 1 + \frac{C}{r}$$

this satisfies the equations

for R_{tt} , R_{rr}

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$R_s =$ Schwarzschild radius = const.

What is R_s ?

For source of mass M recall in weak field limit

$$g_{00} = -(1 + 2\Phi)$$

Newtonian potential.

$$\Rightarrow g_{00} = -\left(1 - \frac{2GM}{r}\right) \quad \Phi = -\frac{GM}{r}$$

$$\Rightarrow \boxed{R_s = 2GM}$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

Schwarzschild radius metric.

As $M \rightarrow 0$ recover the Minkowski metric of flat space.

As $r \rightarrow \infty$ also recover flat space (asymptotic flatness)

Birkhoff's Theorem

The Schwarzschild metric is the unique vacuum solution with spherical symmetry.

Proof: See Carroll §5.2.

\Rightarrow Any spherically symmetric vacuum ^{metric} possesses a timelike Killing vector ∂_t
 \rightarrow called stationary metric

there's No time dependent solutions.

$$ds^2 = g_{00}(\vec{x}) dt^2 + g_{0i}(\vec{x})(dt dx^i + dx^i dt) + g_{ij} dx^i dx^j.$$

rotating star.

A metric is static if timelike Killing vector is orthogonal to hypersurfaces ($t = \text{const}$, hypersurface).

$$\rightarrow ds^2 = g_{00}(\vec{x}) dt^2 + g_{ij}(\vec{x}) dx^i dx^j$$

e.g. nonrotating stars, blackholes \rightarrow static
 rotating systems \rightarrow stationary.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$r=0$

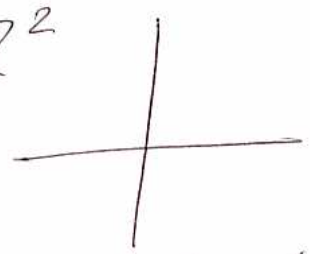
$r = 2GM$

?

singularity?

⊗

e.g. \mathbb{R}^2



$$ds^2 = dr^2 + r^2 d\theta^2$$

$$r=0$$

$$g^{00} = \frac{1}{r^2}$$

coordinate singularity :
results from breakdown of coord. system.

~~R^0~~ scalar
 ~~$R_{\mu\nu}$~~ $R_{\mu\nu}$ $R_{\mu\nu}$ $R_{\mu\nu}$
 R , $R_{\mu\nu}$ $R^{\mu\nu}$, ...

A true singularity "signals the geometry is infinite".

For Sch. metric $R^{\mu\nu} g_{\mu\nu}$ $R_{\mu\nu} g_{\mu\nu} = \frac{48G^2 M^2}{r^2}$

$r=0 \Rightarrow$ true ^{classical} singularity

$r = 2G_1 M$ - coordinate singularity.

$$M_{\odot} = 2 \times 10^{30} \text{ kg}$$

Solar radius $R_{\odot} = 10^6 G_1 M_{\odot} \gg 2G_1 M_{\odot}$

$$2 \cdot 10^{-11} \cdot 10^{30}$$