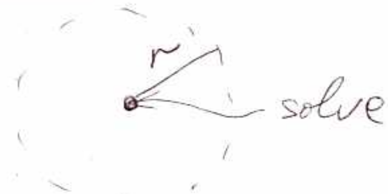
Phys 8501 Lecture 15 (T 10.26.2004)

The Schwarzschild Solution

Einstein's equation in a vacuum:



$$R_{\mu\nu} = 0$$

$$\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \right)$$ You can take Trace and show $R = 0$

Look for solutions of $g_{\mu\nu}$ that are static and spherically symmetric.

static $\Rightarrow$ $g_{\mu\nu}$ independent of t.
(and no crossterms: $dx\,dt$ → assume symmetry $t \to -dt$)

spherically symmetric
$\Rightarrow$ use polar coordinats
(t, r, θ, φ)

Start with ansatz

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 \underbrace{d\Omega^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$$

Define $r' = e^{\gamma(r)} r$

$$dr' = e^{\gamma} dr + r e^{\gamma} \frac{d\gamma}{dr} dr = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma} dr$$

$$\Rightarrow ds^2 = -e^{2\alpha(r)} dt^2 + \underbrace{\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta - 2\gamma}}_{e^{2\beta'(r)}} dr'^2 + r'^2 d\Omega^2$$

$$\Rightarrow ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Need to solve for $\alpha(r)$ and $\beta(r)$

Substituting $g_{\mu\nu}$ into $\Gamma^{\mu}{}_{\rho\sigma}$ and $R^{\rho}{}_{\mu\nu\lambda}$, obtain

$$0 = R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right]$$

$$0 = R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta$$

$$0 = R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$$

$$0 = R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr}$$

$$= \frac{2}{r}\partial_r(\alpha+\beta) \Rightarrow \alpha + \beta = \text{const} \equiv c$$

$$ds^2 = -e^{-2\beta}e^{2c}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \ldots$$

$$t' = e^{c}t$$

Absorb c by rescaling $t \to e^{-c}t$

$$\alpha = -\beta$$

Next $R_{\theta\theta} = 0$

$$\Rightarrow e^{2\alpha}\left[r(-2)\partial_r\alpha - 1\right] + 1 = 0$$

$$\frac{d(re^{2\alpha})}{dr} = 1$$

$$\Rightarrow r e^{2\alpha} = r + C$$

$$\Rightarrow e^{2\alpha} = 1 + \frac{C}{r}$$ this satisfies the equations for R_{tt}, R_{rr}

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

R_s = Schwarzschild radius = const.

What is R_s?

For source of mass M recall in weak field limit

$$g_{00} = -(1 + 2\underbrace{\Phi}_{\text{potential}})$$

~~Newtonian pot~~ potential. $\Phi = -\frac{GM}{r}$

$$\Rightarrow g_{00} = -\left(1 - 2\frac{GM}{r}\right)$$

$$\Rightarrow \boxed{R_s = 2GM}$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

Schwarzschild ~~radius~~ metric.

As $M \to 0$ recover the Minkowski metric of flat space.

As $r \to \infty$ also recover flat space (asymptotic flatness)

Birkhoff's Theorem

The Schwarzschild metric is the unique vacuum solution with spherical symmetry.

Proof: See Carroll §5.2.

$\Rightarrow$ Any spherically symmetric vacuum metric possesses a timelike killing vector ∂_t
$\rightarrow$ called stationary metric

There's No time dependent solutions.

$$ds^2 = g_{00}(\vec{x})dt^2 + g_{0i}(\vec{x})(dt\,dx^i + dx^i dt) + g_{ij}\,dx^i dx^j.$$

rotating star.

A metric is static if timelike killing vector is orthogonal to hypersurfaces (t = const, hypersurface).

$$\rightarrow ds^2 = g_{00}(\vec{x})dt^2 + g_{ij}(\vec{x})dx^i dx^j$$

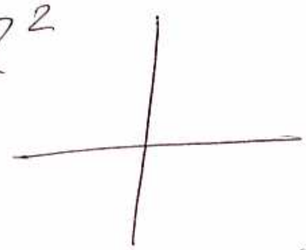
e.g. nonrotating stars, blackholes $\rightarrow$ static
rotating systems $\rightarrow$ stationary.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega$$

$r = 0$ $\quad r = 2GM$? singularity ?

e.g.

e.g. $\mathbb{R}^2$



$$ds^2 = dr^2 + r^2 d\theta^2$$

$r = 0 \qquad g^{\theta\theta} = \frac{1}{r^2}$

<u>coordinate singularity</u>:
results from breakdown of coord. system.

scalar
$\cancel{R^{\rho}{}_{\mu\nu\sigma}}\ R_{\rho\mu\nu\sigma}\ R^{\rho\mu\nu\sigma}$

$R,\ R_{\mu\nu} R^{\mu\nu}, \dots$

A true singularity signals the geometry is "infinite":

For Sch. metric $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^2}$

$r = 0 \Rightarrow$ true classical singularity

$r = 2GM$ — coordinate singularity.

$M_{\odot} = 2 \times 10^{30}$ kg

Sch solar radius $R_{\odot} = 10^6 G M_{\odot} \gg 2 G M_{\odot}$

$2\,10^{-11} \cdot 10^{30}$