Phys 8501 Lecture 13 (F 10.22.04)

Lagrangian formulation

What is the Lagrangian for GR?

In general

$$S = \int \mathcal{L}(\varphi^i, \nabla_\mu \varphi^i)\, d^n x$$

$$= \int \hat{\mathcal{L}} \underbrace{\sqrt{g}\, d^n x}^{\varepsilon} \quad (\text{c.f. } \int \varphi \epsilon = \int \varphi \sqrt{-g}\, d^n x)$$

$$\mathcal{L} = \sqrt{-g}\, \hat{\mathcal{L}}$$

Hilbert proposed:

$$\boxed{S_H = \int d^n x \sqrt{-g}\, R}$$

Einstein-Hilbert action

$R = g^{\mu\nu} R_{\mu\nu}$

variation gives Einstein equation

Whant to recover Einstein's equations from varying S_H under small variations $\delta g_{\mu\nu}$.

$$\delta S_H = \int d^n x\, \delta(\sqrt{-g})\, R + \int d^n x \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \int d^n x \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} =$$

$$= \int d^n x \sqrt{-g}\, \underbrace{(R_{\mu\nu} + \dots\dots)}_{\text{has to vanish}}\, \delta g^{\mu\nu}$$

$\delta R_{\mu\nu}$ $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$; $\Gamma^{\rho}_{\mu\nu} \to \Gamma^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}$

$\Rightarrow \delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}{}_{\lambda\mu})$

$\delta\Gamma^{\sigma}_{\mu\nu} =$ see book.

$$\int d^4x \sqrt{-g}\,\delta R_{\mu\nu} g^{\mu\nu} = \int d^4x \sqrt{-g}\,\nabla_\sigma \left[g_{\mu\nu} \nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda}) \right]$$

$$= \int_{\partial\Sigma} d^{n-1}x \sqrt{-\gamma}\, n_\sigma \Big(g_{\mu\nu} \underbrace{\nabla^\sigma(\delta g^{\mu\nu})}_{\to 0} - \underbrace{\nabla_\lambda(\delta g^{\sigma\lambda})}_{\to 0} \Big)$$

$\uparrow$ induced metric

$= 0$

$\delta(\sqrt{-g})$ Need the variation of the det g:

Use $\ln(\det M) = \mathrm{Tr}(\ln M)$ arbitrary M

$$\delta \ln(\det M) = \frac{1}{\det M} \cdot \delta \det M = \mathrm{Tr}\left(\frac{1}{M}\delta M\right)$$

Put ~~$g = \det$~~ $M = g_{\mu\nu}$, $g = \det g_{\mu\nu}$

$$\delta g = g\,\mathrm{Tr}(g^{\mu\lambda}\delta g_{\lambda\nu}) =$$

$$= g\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta(\overbrace{g^{\mu\nu}g_{\nu\lambda}}^{\delta^\mu_\lambda}) = 0 = \delta g^{\mu\nu}\, g_{\nu\lambda} + g^{\mu\nu}\delta g_{\nu\lambda} = 0$$

$$\delta g = -g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta(\sqrt{-g}) = \frac{1}{2}(-g)^{-1/2}(-1)\,\delta g =$$

$$= \frac{-1}{2\sqrt{-g}}(-g)\, g_{\mu\nu}\, \delta g^{\mu\nu} =$$

$$= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta S = \int d^4x \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\delta g^{\mu\nu}$$

$8\pi G T_{\mu\nu}$

$$\frac{\delta S_H}{\sqrt{-g}\,\delta g^{\mu\nu}} = 0 \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0}$$

Einstein's equations in a vacuum!

To include matter consider

$$S = \frac{1}{16\pi G} S_H + S_M$$

$$S_M = \int d^4x \sqrt{-g}\, \frac{1}{2} (\nabla_\mu \phi)(\nabla_\nu \phi)\, g^{\mu\nu}$$

$\int d^4x \cancel{\sqrt{-g}}^{\,1} \frac{1}{2} \partial_\mu \phi \partial_\nu \phi\, \eta^{\mu\nu}$ in Minkowski.

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{total}}{\delta g^{\mu\nu}} = 0 \implies$$

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \overbrace{\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}}^{\frac{1}{2}T_{\mu\nu}} = 0.$$

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}}$$

The $T_{\mu\nu}$ definition is symmetric (because $g^{\mu\nu}$ is) $(0,2)$ tensor.
and conserved.

Apply
∇_μ to the Einstein's equation

$$\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

$$\nabla^\mu G_{\mu\nu} = 0.$$

Properties of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$

- 2nd order d.e. for metrics $g_{\mu\nu}$

$\nabla^2 \phi = 4 G \rho$

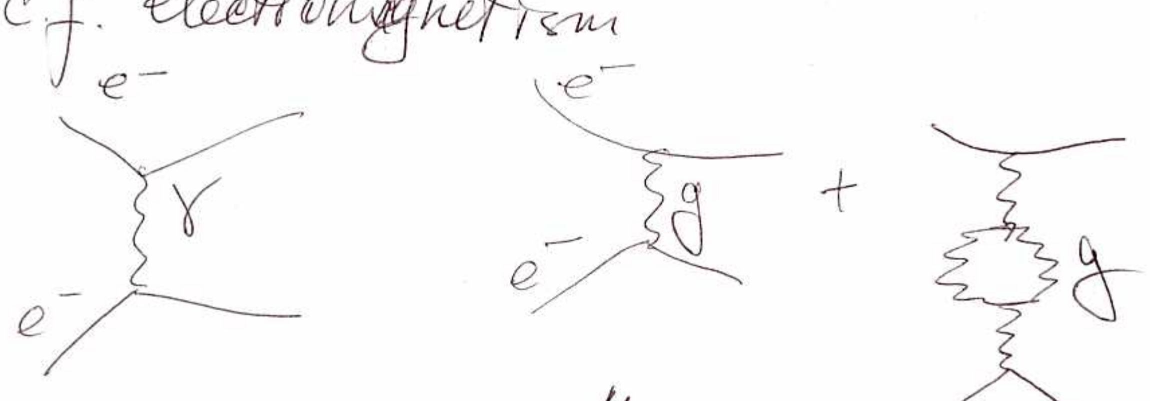
$\vec{a} = \nabla \phi \quad \nabla \vec{a} = 4\pi G \rho \rightarrow \nabla \vec{E} = \frac{q}{\epsilon_0}$

- 10 equations (in 4 dimensions) only 6 independent.

$\nabla^\mu G_{\mu\nu} = 0$ (4 eqn)

- Equations are nonlinear

c.f. electromagnetism



- Let $\Theta = \nabla_\mu U^\mu$ = expansion parameter (of small ball of test particles with velocity U^μ)

then $\frac{d\Theta}{dt} = -4\pi G (\rho + p_x + p_y + p_z)$ pressure

fluid $T_{\mu\nu} = (\rho \; -p \; -p \; -p)$ equivalent to Einstein's equation

see Baez gr-qc/0103044

$\rho + p_i < 0 \Rightarrow$ cosmological constant (antigravity)

- Quantum Gravity $\longrightarrow$

GR is a <u>classical</u> field theory.

At very short distances $\longrightarrow$ quantising gravity is problematic

gravity is nonrenormalisable.

Dimensionfull parameters: $\hbar, c, G$

Plank Mass (Energy) $m_p = \left(\frac{\hbar c}{G}\right)^{1/2}$

$$E_p = m_p c^2 = 10^{19}\,\text{GeV}$$

Plank length $\ell_p = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-33}\,\text{cm}$

current experiment: $10^3\,\text{GeV} \rightarrow 10^{-16}\,\text{cm}$