

Lagrangian formulation

What is the Lagrangian for GR?

In general

$$S = \int \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) d^n x$$

$$= \int \hat{\mathcal{L}} \sqrt{-g} d^n x \quad (\text{c.f. } \int \mathcal{P} \epsilon = \int \mathcal{P} \sqrt{-g} d^n x)$$

$$\mathcal{L} = \sqrt{-g} \hat{\mathcal{L}}$$

Milbert proposed:

$$S_H = \int d^n x \sqrt{-g} R$$

Einstein-Milbert action

$$R = g^{\mu\nu} R_{\mu\nu}$$

variation gives Einstein equation

Want to recover Einstein's equations from varying  $S_H$  under small variations  $\delta g^{\mu\nu}$ .

$$\delta S_H = \int d^n x \delta(\sqrt{-g}) R + \int d^n x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^n x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} =$$

$$= \int d^n x \sqrt{-g} \underbrace{(R_{\mu\nu} + \dots)}_{\text{has to vanish}} \delta g^{\mu\nu}$$

$$\delta R_{\mu\nu} \approx R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}; \quad \Gamma_{\mu\nu}^{\rho} \rightarrow \Gamma_{\mu\nu}^{\rho} + \delta\Gamma_{\mu\nu}^{\rho}$$

$$\Rightarrow \delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma_{\nu\mu}^{\rho}) - \nabla_{\nu}(\delta\Gamma_{\lambda\mu}^{\rho})$$

$\delta\Gamma_{\mu\nu}^{\rho}$  = see book.

$$\int d^4x \sqrt{-g} \delta R_{\mu\nu} g^{\mu\nu} = \int d^4x \sqrt{-g} \nabla_{\lambda} [g^{\mu\nu} \nabla^{\lambda}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\mu\lambda})]$$

$$= \int_{\partial\Sigma} d^3x \sqrt{-g} n_{\lambda} (g^{\mu\nu} \nabla^{\lambda}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\mu\lambda}))$$

↑  
induced metric

= 0

$\delta(\sqrt{-g})$  Need the variation of the det-g:

Use  $\ln(\det M) = \text{Tr}(\ln M)$  arbitrary M

$$\delta \ln(\det M) = \frac{1}{\det M} \cdot \delta \det M = \text{Tr} \left( \frac{1}{M} \delta M \right)$$

Put  $g = \det M = g_{\mu\nu}$ ,  $g = \det g_{\mu\nu}$

$$\delta g = g \text{Tr} (g^{\mu\nu} \delta g_{\mu\nu}) =$$

$$= g g^{\mu\nu} \delta g_{\mu\nu}$$

$$\delta(g^{\mu\nu} g_{\nu\lambda}) = 0 = \delta g^{\mu\nu} g_{\nu\lambda} + g^{\mu\nu} \delta g_{\nu\lambda} = 0$$

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}$$

$$\begin{aligned} \delta(\sqrt{-g}) &= \frac{1}{2} (-g)^{-1/2} (-1) \delta g = \\ &= -\frac{1}{2\sqrt{-g}} (-g) g_{\mu\nu} \delta g^{\mu\nu} = \\ &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \end{aligned}$$

$$\delta S = \int d^4x \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}$$

$8\pi G T_{\mu\nu}$

$$\frac{\delta S_H}{\sqrt{-g} \delta g^{\mu\nu}} = 0 \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0}$$

Einstein's equations in a vacuum!

To include matter consider

$$S = \frac{1}{16\pi G} S_H + S_M$$

$$S_M = \int d^4x \sqrt{-g} \frac{1}{2} (\nabla_\mu \phi)(\nabla^\mu \phi) g^{\mu\nu}$$

$\int d^4x \sqrt{-g} \frac{1}{2} \partial_\mu \phi \partial^\mu \phi g^{\mu\nu}$  in Minkowski.

$$\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{total}}}{\delta g^{\mu\nu}} = 0 \Rightarrow \frac{1}{16\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0.$$

$$T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$

The  $T_{\mu\nu}$  definition is symmetric (because  $g_{\mu\nu}$  is)  $(0,2)$  tensor.  
and conserved.

Apply  $\nabla_{\mu}$  to the Einstein's equation

$$\nabla^{\mu} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)$$

$$\nabla^{\mu} G_{\mu\nu} = 0.$$

Properties of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- 2nd order d.e. for metrics  $g_{\mu\nu}$

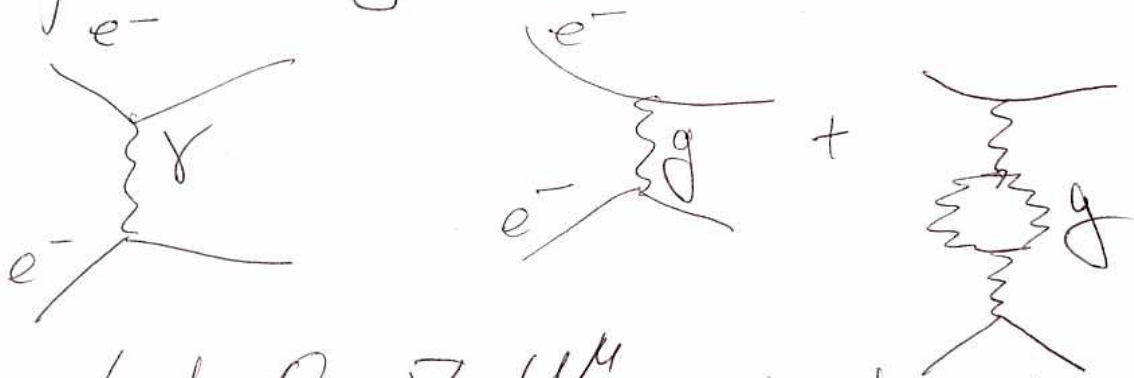
$$\nabla^2 \phi = 4\pi G \rho$$

$$\vec{a} = \nabla \phi \quad \nabla \vec{a} = 4\pi G \rho \rightarrow \nabla \vec{E} = \frac{q}{\epsilon_0}$$

- 10 equations (in 4 dimensions) only 6 independent.

$$\nabla^\mu G_{\mu\nu} = 0 \quad (4 \text{ eqn})$$

- Equations are nonlinear  
c.f. electromagnetism



- Let  $\Theta = \nabla_\mu U^\mu =$  expansion parameter (of small ball of test particles with velocity  $u^\mu$ )

fluid  $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p_x & & \\ & & p_y & \\ & & & p_z \end{pmatrix}$  pressure  
 Then  $\frac{d\Theta}{dt} = -4\pi G (\rho + p_x + p_y + p_z)$  equivalent to Einstein's equation  
 see Baer gr-qc/0103044  
 $\rho + p_i < 0 \Rightarrow$  cosmological constant (antigravity)

• Quantum Gravity →

GR is a classical field theory.

At very short distances → quantizing gravity is problematic  
gravity is nonrenormalizable.

Dimensionful parameters:  $\hbar, c, G$

$$\text{Planck Mass (Energy)} \quad m_p = \left( \frac{\hbar c}{G} \right)^{1/2}$$

$$E_p = m_p c^2 = 10^{19} \text{ GeV}$$

$$\text{Planck length} \quad l_p = \left( \frac{\hbar G}{c^3} \right)^{1/2} = 1.62 \times 10^{-33} \text{ cm}$$

current experiment:  $10^3 \text{ GeV} \rightarrow 10^{-16} \text{ cm}$