Phys 8501 Lecture 11 (T 10.12.04)

Symmetries & Killing vectors

What are symmetries of metric?
Known as isometries

Recall geodesic equation (particle on timelike path).

$$p^\lambda \nabla_\lambda p^\mu = 0$$

$$\Rightarrow p^\lambda \partial_\lambda p_\mu = \Gamma^\sigma{}_{\lambda\mu} p^\lambda p_\sigma$$

$$\Rightarrow m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = \frac{1}{2} g^{\sigma\nu} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) \cdot p^\lambda p_\sigma$$

use chain rule

$$\Rightarrow m \frac{dp_\mu}{d\tau} = \frac{1}{2} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p^\nu$$

$$\Rightarrow m \frac{dp_\mu}{d\tau} = \frac{1}{2} (\partial_\mu g_{\nu\lambda}) p^\lambda p^\nu$$

fix coords:

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \Rightarrow \frac{dp_{\sigma_*}}{d\tau} = 0$$

$$\Rightarrow p_{\sigma_*} \text{ conserved quantity.}$$

11

Manifestly covariant formulation

Let $K = \partial_{\sigma_*} \implies K = K^\mu \partial_\mu$

$\implies K^\mu = \delta^\mu_{\sigma_*}$

"K^μ generates the isometry"

$p_{\sigma_*} = K_\mu p^\mu = K^\mu p_\mu$

$$p^\mu \nabla_\mu (\underbrace{K_\nu p^\nu}_{p_{\sigma_*}}) = \cancel{p^\mu K_\nu \nabla_\mu p^\nu} + p^\mu p^\nu \nabla_\mu K_\nu$$

$\to 0$ (geodesic)

$$= p^\mu p^\nu \nabla_\mu K_\nu = p^\mu p^\nu \nabla_{(\mu} K_{\nu)}$$

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0} \implies$$

$$p^\mu \nabla_\mu (K_\nu p^\nu) = 0 \quad \text{or} \quad \frac{dp_{\sigma_*}}{d\tau} = 0$$

$\to$ Killing's equation

Solutions are ~~called~~ known as Killing vectors

Example $\mathbb{R}^3$ $ds^2 = dx^2 + dy^2 + dz^2$

Killing vectors: $\partial_x, \partial_y, \partial_z$

there are also Killing vectors associated with ~~rot~~ rotation:

$-y\partial_x + x\partial_y, \quad z\partial_x - x\partial_z, \quad -z\partial_y + y\partial_x$

In Minkowski space $\to$ 10 Killing vectors

HW: forget about # 4.

$R^{\rho}{}_{\sigma\mu\nu} \xrightarrow{g_{\mu\nu} \to \omega^2 g_{\mu\nu}}$ $\omega^{-1}\partial_\mu \omega = \partial_\mu \ln \omega$

$\sim \longrightarrow \partial_\mu \Gamma$ $\qquad \partial_\lambda (g^{\lambda\rho} g_{\mu\nu})$

$\Gamma_{\mu\nu}{}^{\rho} = \Gamma^{\rho}_{\mu\nu} + f(\omega)\partial_\varphi$ $\qquad$ does not depend on ω

Maximally symmetric spaces

Consider $\mathbb{R}^n$ flat metric

What are isometries of $\mathbb{R}^n$?

$p \in \mathbb{R}^n$. translation along n axes $\rightarrow$ n isometries

rotations about $p \rightarrow \frac{1}{2}n(n-1)$ isometries

Total: $\frac{1}{2}n(n+1)$

An n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors is (known as) <u>maximally symmetric space</u>.

e.g. $\mathbb{R}^n$, S^n

The curvature of maximally symmetric space is the same everywhere.

$\Rightarrow$ curvature tensor is the same in all directions

Choose locally inertial coords at $p \in M$

$g_{\hat\mu\hat\nu} = \eta_{\hat\mu\hat\nu}$

What is $R_{\hat\rho\hat\sigma\hat\mu\hat\nu}$?

Symmetry properties of Riemann tensors

$R_{\hat\rho\hat\sigma\hat\mu\hat\nu} \Rightarrow$

$$R_{\hat\rho\hat\sigma\hat\mu\hat\nu} \propto g_{\hat\rho\hat\mu} g_{\hat\sigma\hat\nu} - g_{\hat\rho\hat\nu} g_{\hat\sigma\hat\mu}$$

$$\Rightarrow R_{\rho\sigma\mu\nu} = c\left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right)$$

this relation is valid throughout the manifold M since space is maximally symmetric.

$$\underbrace{\underbrace{\underbrace{g^{\sigma\nu} g^{\mu\rho} R_{\rho\sigma\mu\nu}}_{R^{\mu}{}_{\sigma\mu\nu}}}_{R_{\sigma\nu}}}_{R} = c\, g^{\sigma\nu} g^{\mu\rho}\left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right)$$

$$R = c\left(\overbrace{\delta^{\mu}{}_{\mu}\delta^{\nu}_{\nu}}^{n^2} - \overbrace{\delta^{\mu}{}_{\nu}\delta^{\nu}_{\mu}}^{\delta^{\mu}_{\mu} = n}\right) = cn(n-1)$$

$$c = \frac{R}{n(n-1)}$$

$$\boxed{R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)}$$

(for maximally symmetric spaces)

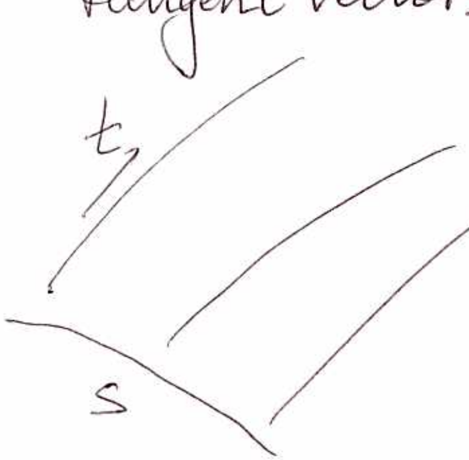
can serve as criteria to check if the space is maximally symmetric.

Example ~~Euclidean~~

	$R=0$	$R>0$	$R<0$
Euclidean	$\mathbb{R}^n$	S^n	hyperboloids H^n
Lorentzian	Minkowski	de Sitter	Anti de Sitter

Geodesic deviation

Consider a set of geodesics $\gamma_s(t)$ with tangent vectors T^μ



$$T^\mu = \frac{\partial x^\mu}{\partial t}$$

deviation vector $S^\mu = \frac{\partial x^\mu}{\partial s}$

Define "relative* velocity of geodesics"

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$$

"relative acceleration":

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu$$

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho \left(T^\sigma \nabla_\sigma S^\mu\right)$$

$$= T^\rho \nabla_\rho \left(S^\sigma \nabla_\sigma T^\mu\right)$$ where $[S,T]=0$

$\Rightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ (see HW #1).

$$= T^\rho (\nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu =$$

$$= T^\rho (\nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma \left(\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu\right)$$

(used the definition of the Riemann tensor)

$$= \cancel{T^\rho (\nabla_\rho S^\sigma) \nabla_\sigma T^\mu} + S^\sigma \nabla_\sigma \overbrace{\left(T^\rho \nabla_\rho T^\mu\right)}^{=0 \text{ (geodesic)}} - \cancel{\left(S^\sigma \nabla_\sigma T^\rho\right) \nabla_\rho T^\mu} + T^\rho S^\sigma R^\mu{}_{\nu\rho\sigma} T^\nu$$

$$\boxed{A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma}$$ geodesic deviation equation

relative acceleration, between geodesics $\propto$ curvature!

Physically corresponds to tidal forces!

Monday 4^{35} pm.

next Friday 6 pm.