Phys 8501 Lecture 10 (M 10.11.2004)

Riemann Tensor

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - \overbrace{T^\lambda{}_{\mu\nu}}^{\text{torsion tensor}} \nabla_\lambda V^\rho$$

$$\Rightarrow \boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}}$$

In general we have

$$[\nabla_\mu \nabla_\nu] X^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = -T^\lambda{}_{\rho\sigma} \nabla_\lambda X^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots - R^\lambda{}_{\nu_1 \rho\sigma} X^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_l} - \cdots$$

Theorem A coordinate system exists in which the metric components are constant $\underbrace{\text{if and only if}}_{\text{iff}}$ the Riemann tensor vanishes

Proof: $(\Rightarrow)$ $\partial_\sigma g_{\mu\nu} = 0 \Rightarrow \Gamma^\rho_{\mu\nu} = 0$

$$\Rightarrow R^\rho{}_{\sigma\mu\nu} = 0.$$

$(\Leftarrow)$ Consider 1-form $\omega = \omega_\mu dx^\mu$

Construct uniquely 1-form along curve $x^\mu(\lambda)$ by parallel transport

$$\frac{dx^\mu}{d\lambda} \nabla_\mu \omega_\nu = 0.$$

But $R^\rho{}_{\sigma\mu\nu} = 0 \Longrightarrow$ parallel transport is path independ

$\Longrightarrow$ there exist a unique 1-form field throughout manifold.

$\xrightarrow[\text{for arbitrary } \frac{dx^\mu}{d\lambda}]{\text{valid}}$ $\nabla_\mu \omega_\nu = 0$

$\Longrightarrow \nabla_{[\mu}\omega_{\nu]} = 0$

$\Longrightarrow \partial_{[\mu}\omega_{\nu]} = 0$

$\Longrightarrow d\omega = 0$

$\xrightarrow{\mathbb{R}^n} \omega = d\alpha$

Now consider set of basis one forms $\hat{\theta}^{(a)}$, $a = 1, \dots n$

$$ds^2(p) = \eta_{ab}\hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}$$

$\xrightarrow[\text{parallel transp.}]{\nabla_\mu g_{\rho\sigma} = 0}$ $ds^2(\text{everywhere}) = \eta_{ab}\hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}$

But $\vec{\theta}^{(a)} = dy^a$
= coordinate basis

$$\Rightarrow \boxed{ds^2 = \eta_{ab}\, dy^a dy^b}$$

<u>Summary</u> If $R^{\rho}{}_{\sigma\mu\nu} = 0$
$\Rightarrow$ metric is flat

$R^{\rho}{}_{\sigma\mu\nu} \neq 0 \Rightarrow$ there is curvature.

Consider locally-inertial coords ($\partial_{\hat{\mu}} g_{\hat{\alpha}\hat{\beta}}|_p = 0$, $\Gamma^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}}|_p = 0$)

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$$

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(p) = \frac{1}{2}\Big(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}}\Big)$$

We see that

(1) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$

(2) $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$

③ $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

others follow

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$

$$R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$$

Number of independent components

Symmetric $n \times n$ matrix $\Rightarrow \frac{n}{2}(n+1)$ comp-s

anty – " – " – " – $\Rightarrow \frac{n}{2}(n-1)$ comps

$$\# \text{ comps} = \frac{1}{2}\left[\frac{n}{2}(n-1)\right]\left[\frac{n}{2}(n-1)+1\right] - \underbrace{\binom{n}{4}}_{\frac{n(n-1)(n-2)(n-3)}{4!}}$$

$$= \frac{1}{12} n^2(n^2-1).$$

$n=4 \Rightarrow 20$ independent components of $R^{\rho}{}_{\sigma\mu\nu}$

Two related tensors

Ricci tensor

$$\boxed{R_{\mu\nu} = \cancel{g^{\lambda\rho}R_{\rho}}\, R^{\lambda}{}_{\mu\lambda\nu}}$$

$R_{\mu\nu}$ is symmetric $R_{\mu\nu} = R_{\nu\mu}$

The trace of the Ricci tensor

Ricci scalar $\boxed{R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}}$

Bianci identity $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$\Rightarrow \nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R$

Weyl tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left[g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right] + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

Under $g_{\mu\nu} \to \omega^2(x) g_{\mu\nu}$

$C^{\rho}{}_{\sigma\mu\nu}$ is invariant.

Einstein tensor

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$] important!

Note that $\nabla^{\mu} G_{\mu\nu} = 0$

Extrinsic curvature

Curvature that arises from embedding in some larger space.

e.g. circle $S^1 \in \mathbb{R}^2$

flat

looks curved in $\mathbb{R}^2$ but intrinsically flat.

In one dimension $R_{\mu\nu\rho\sigma} = 0$

torus $S^1 \times S^1$

intrinsically flat

$ds^2 = dx^2 + dy^2$.

Riemann curvature is an intrinsic property of the manifold

Example: sphere S^2

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

In 2 dims have $\frac{1}{12} 2^2(2^2-1) = 1$

Obtain $R_{\theta\phi\theta\phi} = a^2 \sin^2\theta$; all others vanish.

Ricci tensor: $R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta$$

$\Rightarrow$ Ricci scalar:

$$R = g^{MN} R_{MN} = g^{\varphi\varphi} R_{\varphi\varphi} + g^{\theta\theta} R_{\theta\theta} =$$

$$= \frac{1}{a^2 \sin^2\theta} \sin^2\theta + \frac{1}{a^2} \cdot 1 = \frac{2}{a^2}$$

$$\boxed{R = \frac{2}{a^2}}$$

nonzero

$\Rightarrow S^2$ is intrinsically curved.