

Phys 8501 Lecture 10 (M 10.11.2004)

Riemann Tensor

torsion tensor

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma - \overset{\text{torsion tensor}}{\Gamma^\lambda_{\mu\nu}} \nabla_\lambda V^\rho$$

$$\Rightarrow \boxed{R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}}$$

In general we have

$$[\nabla_\mu, \nabla_\nu] X^{M_1 \dots M_k}{}_{P_1 \dots P_k} = -T^\lambda_{\rho\sigma} \nabla_\lambda X^{M_1 \dots M_k}{}_{P_1 \dots P_k} \\ + R^{M_1}{}_{\lambda\rho\sigma} X^{\lambda M_2 \dots M_k}{}_{P_1 \dots P_k} + \dots \\ \neq -R^\lambda{}_{\rho\sigma} X^{M_1 \dots M_k}{}_{\lambda P_2 \dots P_k} \neq \dots$$

Theorem A coordinate system exists in which the metric components are constant if and only if the Riemann tensor vanishes iff

Proof: $(\Rightarrow) \partial_\sigma g_{\mu\nu} = 0 \Rightarrow \Gamma^\rho_{\mu\nu} = 0$
 $\Rightarrow R^\rho_{\sigma\mu\nu} = 0.$

(\Leftarrow) Consider 1-form $\omega = \omega_\mu dx^\mu$

Construct uniquely 1-form along curve $x^\mu(\lambda)$
by parallel transport

$$\frac{dx^\mu}{d\lambda} \nabla_\mu \omega_\nu = 0.$$

but $R^{\rho}_{\sigma\mu\nu} = 0 \Rightarrow$ parallel transport is path independent
 \Rightarrow there exist a unique 1-form field throughout
manifold.

valid
for arbitrary

$$\frac{dx^\mu}{d\lambda} \nabla_\mu \omega_\nu = 0$$

$$\Rightarrow \nabla_\mu [\frac{dx^\mu}{d\lambda} \omega_\nu] = 0$$

$$\Rightarrow \partial_\mu [\frac{dx^\mu}{d\lambda} \omega_\nu] = 0$$

$$\Rightarrow d\omega = 0$$

$$\xrightarrow{\mathbb{R}^n} \omega = dd$$

Now consider set of basis one forms $\hat{\theta}^a$,

$$a = 1, \dots, n$$

$$ds^2(p) = \eta_{ab} \hat{\theta}^a \otimes \hat{\theta}^b$$

$$\nabla_\mu \eta_{ab} = 0$$

parallel transport

$$ds^2(\text{everywhere}) = \eta_{ab} \hat{\theta}^a \otimes \hat{\theta}^b$$

to

but $\partial^a = dy^a$
 = coordinate basis

$$\Rightarrow ds^2 = \eta_{ab} dy^a dy^b$$

Summary If $R^{\rho}_{\sigma\mu\nu} = 0$
 \Rightarrow metric is flat

$R^{\rho}_{\sigma\mu\nu} \neq 0 \Rightarrow$ there is curvature.

Consider locally-inertial coords ($\partial_{\mu}^{\alpha} g^{\beta\gamma} = 0$
 $\Gamma^{\lambda}_{\mu\nu} = 0$)

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}_{\sigma\mu\nu}$$

$$R^{\rho\sigma\mu\nu}(\rho) = \frac{1}{2} (\partial_{\mu}^{\rho} \partial_{\sigma}^{\nu} g^{\lambda\lambda} - \partial_{\mu}^{\rho} \partial_{\sigma}^{\lambda} g^{\lambda\nu} - \partial_{\sigma}^{\rho} \partial_{\mu}^{\lambda} g^{\lambda\nu} + \partial_{\sigma}^{\rho} \partial_{\mu}^{\nu} g^{\lambda\lambda})$$

We see that

① $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$

② $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$

$$\textcircled{3} R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

others follow

$$R_{\rho\sigma\mu\nu} + R_{\sigma\mu\rho\nu} + R_{\rho\nu\sigma\mu} = 0$$

$$R_{\rho}[\sigma\mu\nu] = 0 \Rightarrow R[\sigma\rho\mu\nu] = 0$$

Number of independent components

Symmetric $n \times n$ matrix $\Rightarrow \frac{n}{2}(n+1)$ comp-s

anti-symmetric $n \times n$ matrix $\Rightarrow \frac{n}{2}(n-1)$ comp-s

$$\# \text{ comp-s} = \frac{1}{2} \left[\frac{n}{2}(n-1) \right] \left[\frac{n}{2}(n-1) + 1 \right] = \binom{n}{4}$$

$$= \frac{1}{12} n^2 (n^2 - 1)$$

$$\frac{n(n-1)(n-2)(n-3)}{4!}$$

$n=4 \Rightarrow 20$ independent components of $R_{\rho\sigma\mu\nu}$

Two related tensors

Ricci tensor

$$R_{\mu\nu} = g^{\lambda\rho} R_{\rho\lambda\mu\nu}$$

$R_{\mu\nu}$ is symmetric $R_{\mu\nu} = R_{\nu\mu}$

The trace of the Ricci tensor

Ricci scalar $\boxed{R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu}}$

Bianchi identity $\nabla_{[\alpha} R_{\beta\gamma]} = 0$

$$\Rightarrow \nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R$$

Weyl tensor

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{2}{n-2} [g_{\alpha[\mu} R_{\nu]\beta} - g_{\beta[\mu} R_{\nu]\alpha}] + \frac{2}{(n-1)(n-2)} g_{\alpha\mu} g_{\beta\nu} R$$

Under $g_{\mu\nu} \rightarrow \omega^2(x) g_{\mu\nu}$
 $C^{\alpha\beta\mu\nu}$ is invariant.

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{important!}$$

Note that $\nabla^{\mu} G_{\mu\nu} = 0$

Extrinsic curvature

Curvature that arises from embedding in some larger space.
e.g. circle $S^1 \subseteq \mathbb{R}^2$



flat

looks curved in \mathbb{R}^2 but intrinsically flat.

In one dimension $R_{\mu\nu\rho\sigma} = 0$

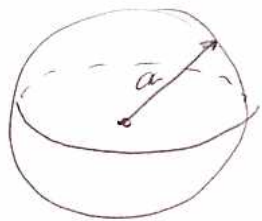
torus $S^1 \times S^1$



intrinsically flat
 $ds^2 = dx^2 + dy^2$.

Riemann curvature is an intrinsic property of the manifold

Example: sphere S^2



$$ds^2 = a^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

In 2 dims have $\frac{1}{12} 2^2 (2^2 - 1) = 1$

Obtain $R_{\theta\phi\theta\phi} = a^2 \sin^2\theta$; all others vanish.

Richi tensor: $R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\theta\phi} = 1$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\phi\theta} = \sin^2\theta$$

\Rightarrow Ricci scalar:

$$R = g^{MN} R_{MN} = g^{\phi\phi} R_{\phi\phi} + g^{\theta\theta} R_{\theta\theta} =$$

$$= \frac{1}{a^2 \sin^2 \theta} \sin^2 \theta + \frac{1}{a^2} \cdot 1 = \frac{2}{a^2}$$

$$\boxed{R = \frac{2}{a^2}}$$

nonzero

$\Rightarrow S^2$ is intrinsically curved.