Phys 8911 Lecture 5 (01.02.2005)

SUSY irreps

2 Casimir ops $C_1 = P^2 = -m^2$

$C_2 = m^+ Y(Y+1) \quad Y = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$= Z_1^2 + Z_2^2 + Z_3^2$

Thus, massive SUSY irreps are classified by (m, Y)

What are the spin states in each SUSY rep?

Massive irreps $m \neq 0$

In rest frame $P_m = (-m, 0, 0, 0)$

$\Rightarrow \{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2m\delta_{\alpha\dot\alpha} = 2m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\{QQ\} = 0 = \{\bar{Q}\bar{Q}\}$

Given any state $|m, Y\rangle$ can define a new state

$|\Omega\rangle = Q_1 Q_2 |m, Y\rangle$

such that $Q_1|\Omega\rangle = 0 = Q_2|\Omega\rangle$

Let $a_\alpha \equiv \frac{1}{\sqrt{2m}} Q_\alpha$

$a_\alpha^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot\alpha}$

where $\{a_\alpha a_\beta^\dagger\} = \delta_{\alpha\beta}$

$\{a, a\} = 0 = \{a^+ a^+\}$ Clifford algebra for 2 fermions

$\{a_\alpha, a^+_\beta\} = \delta_{\alpha\beta}$

then $|\Omega\rangle$ is annihilated by $a_\alpha|\Omega\rangle = 0$

$\Rightarrow |\Omega\rangle$ is a vacuum state

Construct reps

$$|\Omega\rangle$$

$$a_1^+|\Omega\rangle$$

$$a_2^+|\Omega\rangle$$

$$\frac{1}{\sqrt{2}} a_1^+ a_2^+ |\Omega\rangle = -\frac{1}{\sqrt{2}} a_2^+ a_1^+ |\Omega\rangle$$

There are a total $4(2Y+1)$ states in massive irrep

$Y_3 = -Y, \; Y$

The spin can be computed by using

$$\left[S_3, \begin{pmatrix} a_2^+ \\ a_1^+ \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} a_2^+ \\ a_1^+ \end{pmatrix}$$

Thus for $|\Omega\rangle \sim (m; Y)$ obtain spin states $S_{\cancel{3}} = Y, \; |Y \pm \frac{1}{2}|, \; Y.$

~~$Y = 0: \; S = \frac{1}{2}$~~

N = 1 massive irreps

$s \backslash Y$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	$\ldots$
0	2	1			
$\frac{1}{2}$	1	2	1		
1		1	2	1	
$\frac{3}{2}$			1	2	
2				1	
	$4 = 2+2$	8 $3+1+4$	12 $6+6$	16 $8+8$	

We cannot construct higher spins coupled with conserved quantity (current)
graviton (spin 2) couples with $T^{\mu\nu}$

$$h_{\mu\nu} T^{\mu\nu} \to \int T^{0\nu} \to P^{\nu}$$

$$h_{\mu\nu\rho} T^{\mu\nu\rho} \to \int T^{0\nu\rho} \to \cancel{P^{\nu\rho}}$$

(Coleman Mandula)

Massless irreps ($m=0$)

$P^m P_m = 0$ choose $P_m = (-E, 0, 0, E)$.
Then $W_a = \lambda P_a$ where λ = helicity, $\lambda = 0, \pm\frac{1}{2}, \pm 1$
$|\text{spin}| = |\lambda|$
Introduce $L_a = W_a - \frac{1}{16} \bar{\sigma}_a^{\dot{\alpha}\alpha} [Q_\alpha \bar{Q}_{\dot{\alpha}}]$

$L^a P_a = 0$
Obtain $L_a = (k + \frac{1}{4}) P_a$ where k = superhelicity $= 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2} \ldots$

Thus, massless SUSY irreps are classified by $\kappa = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$

"superspin" $= |\kappa|$

What helicities are allowed for each superhelicity?

In massless frame

$$\{Q_\alpha \bar{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha} P_m \quad \{Q_1 \bar{Q}_{\dot 1}\} = 2(\underbrace{\sigma^0_{1\dot1}}_{-1}(-E) + \underbrace{\sigma^3_{1\dot1}}_{1} E) = 4E$$

$$\{Q_2, \bar{Q}_{\dot 2}\} = 2(\underbrace{\sigma^0_{2\dot2}}_{-1}(-E) + \underbrace{\sigma^3_{2\dot2}}_{-1} E) = 0$$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $a \equiv \frac{1}{\sqrt{2E}} Q_1 \quad a^+ \equiv \frac{1}{\sqrt{2E}} \bar{Q}_{\dot 1}$

then $\{a a^+\} = 1 \quad \{a, a\} = \{a^+ a^+\} = 0$

Clifford algebra for 1 fermion.

Consider vacuum $|\Omega\rangle$ defined by $a|\Omega\rangle = 0$

Now $0 = \langle \Omega | \{ Q_2 \bar{Q}_{\dot{2}} \} | \Omega \rangle =$

$$= \langle \Omega | Q_2 (Q_2)^* + Q_2^* Q_2 | \Omega \rangle$$

$$= | Q_2^* | \Omega \rangle |^2 + | Q_2 | \Omega \rangle |^2$$

$$\Rightarrow Q_2^* | \Omega \rangle = 0 = | Q_2 | \Omega \rangle$$

$$\Rightarrow Q_2 = Q_2^* = 0.$$

helicity

Only possible reps $|k\rangle = |\Omega\rangle$ $\quad \lambda$

$a^+ |\Omega\rangle \quad \lambda + \frac{1}{2}$

Lorentz-invariant theory $\Rightarrow$

CPT Symmetry. $\Rightarrow$

$\lambda = -\lambda$

$\lambda + \frac{1}{2} \rightarrow -(\lambda + \frac{1}{2})$

Thus for $|\Omega\rangle = |m=0, k\rangle$ obtain 2 massless particles of helicity $\lambda = k, k + \frac{1}{2}$ $\quad k > 0$

$k, -(|k| + \frac{1}{2})$ $\quad k < 0$

N=1 massless irreps

spin $\lvert\lambda\rvert$ \ $\lvert k\rvert$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
0	1			
$\frac{1}{2}$	1	1		
1		1	1	
$\frac{3}{2}$			1	1
2	=2+2	4=2+2	=2+2	2+2 1

total # of states = 4