Phys 8911 Lecture 4 (M 31.01.2005)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{1}{2}\bar{\Psi}_M\gamma^\mu\partial_\mu\Psi_M +$$

+ interactions $(\sim \xi \varphi_i \bar{\Psi}\Psi)$

$$S_{\mu\alpha} = \partial_\rho(\varphi_1 - i\varphi_2)(\gamma^\rho\gamma_\mu\Psi_M)_\alpha$$

fermionic current

$$\partial^\mu S_{\mu\alpha} = 0$$

$Q_\alpha = \int d^3x\, S_{0\alpha}$ = conserved charge

Avoids C-M theorem with spinor generators!

Generalisation was done by Gelfand & Likhtman 1971

→ algebra now contains anticommutation relations

Known as superalgebra (or graded Lie algebra)

Let $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ denote $\left\{\begin{matrix}\text{bosonic}\\ \text{fermionic}\end{matrix}\right\}$ operators $\left\{\begin{matrix}P_a, M_{ab}\\ Q_\alpha\end{matrix}\right\}$

[even, even] = even ← Poincaré algebra

{odd, odd} = even

[even, odd] = odd

} superalgebra

$$t_a t_b - (-1)^{n_a n_b} t_b t_a = i C_{ab}{}^c t_c$$

n_a = grading $= \left\{\begin{matrix}+1 & \text{fermionic}\\ -1 & \text{boson}\end{matrix}\right.$

$$T(\alpha)T(\beta) = T(f(\alpha,\beta))$$

Fermionic charges extend spacetime symmetries!

In 1975 Haag, Lopuszanski and Sohnius

(Nucl. Phys. B88 (1975) 257) proved that supersymmetry is the only additional symmetry allowed by including spinors
↳ classified superalgebra

For <u>one</u> spinor generator Q_α obtain (HLS)

$[P_m, Q_\alpha] = 0 = [P_m, \bar{Q}_{\dot\alpha}]$ (since spacetime translation do not affect spinors)

$[M_{ab}, Q_\alpha] = (\sigma_{ab})_\alpha{}^\beta Q_\beta$

$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha} P_m$

$\{Q_\alpha, Q_\alpha\} = 0 = \{\bar{Q}_{\dot\alpha}\, \bar{Q}_{\dot\alpha}\}$

Known $N=1$ supersymmetry algebra (simple supersymmetry).

$e^{i b^m P_m + i \zeta^{ab} M_{ab} + \theta^\alpha Q_\alpha}$

$(e^{i b^m P_m})(e^{i b'^m P_m}) = e^{i \ldots}$

$= 1 + i b^m P_m + \ldots$

$[J_i, J_j] = i \varepsilon_{ijk} J^k$

$SO(3) \sim SU(2)$

Note: Can have internal $U(1)$ symmetry, R which does not commute with supersymmetry generators

R-symmetry

$$[Q_\alpha, R] = r Q_\alpha$$
$$[\bar{Q}_{\dot\alpha}, R] = -r \bar{Q}_{\dot\alpha}$$

(r ↑ R-charge)

Consequences

① The energy of every non-vacuum state is positive definite!

Proof: $\frac{1}{2} \bar\sigma^{n\dot\beta\alpha} \{Q_\alpha \bar{Q}_{\dot\beta}\} = 2 \underbrace{\bar\sigma^{n\dot\beta\alpha} \sigma^m{}_{\alpha\dot\beta}}_{\mathrm{Tr}\,\bar\sigma^n \sigma^m = 2\eta^{nm}} p_m = 4P^n$

$\therefore n=0$ $\quad 4\langle\psi|p^0|\psi\rangle = \langle\psi|Q_1 \bar{Q}_{\dot 1} + \bar{Q}_{\dot 1} Q_1 + \bar{Q}_{\dot 2} Q_2 + Q_2 \bar{Q}_{\dot 2}|\psi\rangle =$

$= \langle\psi|Q_1 (Q_1)^* + (Q_1)^* Q_1 + (Q_2)^* Q_2 + Q_2 (Q_2)^*|\psi\rangle =$

$= \langle\psi|Q_\alpha (Q_\alpha)^* + (Q_\alpha)^* Q_\alpha|\psi\rangle \geq 0!$

Also $\langle 0|p^0|0\rangle = 0 \Longleftrightarrow Q_\alpha|0\rangle = 0$ — vacuum is supersymmetric

$e^{i\theta^\alpha Q_\alpha}|0\rangle = |0\rangle$

$(1+i\theta^\alpha Q_\alpha)|0\rangle = |0\rangle.$

The energy of the SUSY vacuum is 0!

<u>Note</u> If $Q_\alpha|0\rangle \neq |0\rangle$ then $\langle 0|p^0|0\rangle > 0$

<u>Consequences</u>

(1.) Every representation has an equal number of bosons and fermion degrees of freedom.

<u>Proof</u>: Introduce $(-1)^{N_F}$ where N_F = fermion number op.

$(-1)^{N_F}|B\rangle = +|B\rangle \qquad (-1)^{N_F}|F\rangle = -1|F\rangle$

$(-1)^{N_F}Q_\alpha = -1\, Q_\alpha(-1)^{N_F}$

Consider $\mathrm{Tr}\left[(-1)^{N_F}\{Q_\alpha \bar{Q}_{\dot\beta}\}\right] =$

$= \mathrm{Tr}\left[(-1)^{N_F}(Q_\alpha \bar{Q}_{\dot\beta} + \bar{Q}_{\dot\beta}Q_\alpha)\right] =$

$= \mathrm{Tr}\left[-Q_\alpha(-1)^{N_F}\bar{Q}_{\dot\beta} + Q_\alpha(-1)^{N_F}\bar{Q}_{\dot\beta}\right] = 0$

$0 = \mathrm{Tr}\left[(-1)^{N_F}\{Q_\alpha \bar{Q}_{\dot\beta}\}\right] = \sum_i \langle i|(-1)^{N_F} 2\sigma^m_{\alpha\dot\beta}\underbrace{p_m|i\rangle}_{p_m|i\rangle} =$

$= 2\sigma^m_{\alpha\dot\beta}p_m \underbrace{\sum_i \langle i|(-1)^{N_F}|i\rangle}_{\mathrm{Tr}(-1)^{N_F}}$

$\Rightarrow \mathrm{Tr}(-1)^{N_F} = 0$

$\Rightarrow$ # bosons = # fermions.

~~Representations of supersymmetry~~

There are two Casimir

1. Mass squared $C_1 = P^2 = P^m P_m$

$[P^2, Q_\alpha] = 0 = [P^2, \bar{Q}_{\dot{\alpha}}]$ since $[P^m, Q_\alpha] = 0$.

$\Rightarrow$ SUSY reps are labeled by mass m. (as they are for the Poincaré group).

2. <u>Generalized Pauli-Lubanski vector</u>

The P-L vector $W^a = \frac{1}{2}\varepsilon^{abcd} P_b M_{cd}$ is <u>no</u> longer a Casimir

Since $[M_{ab}, Q_\alpha] \neq 0 \Rightarrow [W^2, Q_\alpha] \neq 0$

SUSY reps, contain particles of <u>different</u> spin!

Instead, consider generalized P-L vector:

$$Z_m = W_m - \frac{1}{8} \bar{\sigma}_m^{\dot{\alpha}\alpha} [Q_\alpha, \bar{Q}_{\dot{\alpha}}]$$

Then $C_2 = (Z^m P_m)^2 - Z^2 P^2$ = "superspin" operator

is a Casimir

(see Buchbinder & Kuzenko)

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Massive irreps ($m \neq 0$).

$P^{\mu} P_{\mu} = -m^2$ choose $P_{\mu} = (-m, 0, 0, 0)$

Then $W_0 = 0$, $W_i = m S_i$, where $[S_i S_j] = i \varepsilon_{ijk} S^k$

see Kuzenko

$Z_m \ldots$

$\Rightarrow C_2 = m^2 (Z_1^2 + Z_2^2 + Z_3^2)$

Since $[Z_a Z_b] = i \varepsilon_{abcd} Z^c P^d \Rightarrow$

$[Z_i Z_j] = i m \varepsilon_{ijk} Z^k$

Obtain $C_2 = m^4 Y(Y+1) \mathbb{1}$

$Y = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

"superspin"