Phys 8911 Lecture 3 (T 25.01.05)

Poincaré group (Π)

$\Pi = (\Lambda, b)$ = Lorentz transf + translations

A representation is given by

$$T(\Lambda, b) = \exp\left[\frac{i}{2} K^{ab} M_{ab} - i b^a P_a\right] = 1 + \delta T$$

where M_{ab} generator of Lorentz transf.

P_a generator of translations.

The generators satisfy

$$[P_a, P_b] = 0$$

$$[M_{ab}, P_c] = i\eta_{ac} P_b - i\eta_{bc} P_a$$

$$[M_{ab} M_{cd}] = i(\eta_{ac} M_{bd} - \eta_{ad} M_{bc} + \eta_{bd} M_{ac} - \eta_{bc} M_{ad})$$

Poincaré algebra

$\cancel{G}$ $g_1 g_2 \in G \longrightarrow g_1 \cdot g_2 = g_3 \quad g_3 \in G$

$$\downarrow$$

$$T(g_1) \cdot T(g_2) = T(g_3)$$

$$(1 + \delta T_1)(1 + \delta T_2) = 1 + \delta T_3$$

$SO(3) \rightarrow [J_i, J_j] = i \epsilon_{ijk} J_k$

Representations of Poincaré group

There are 2 Casimir operators (commute with M_{ab}, P_a).

$$C_1 = -P^a P_a$$

$$C_2 = W^a W_a$$

$$W_a = \frac{1}{2} \varepsilon_{abcd} M^{bc} P^d \quad \text{Pauli-Lubanski vector}$$

Massive irreps (irreducible representations) $m > 0$

$$P^a P_a = -m^2 = -E^2 + \vec{p}^{\,2}$$

Choose $P_a = (-m, 0, 0, 0)$

then $W_a = (0, m \underbrace{S_i}_{= \frac{1}{2}\varepsilon_{ijk} M^{jk}})$ rotation generator

Where $[S_i, S_j] = i \varepsilon_{ijk} S_k$

(M^{ik} ← boosts).

$$S_1^2 + S_2^2 + S_3^2 = s(s+1)\mathbb{1} \quad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$\Rightarrow W^a W_a = m^2 s(s+1)\mathbb{1}$$

∴ Π is classified by the mass m and spin s.

Massless irreps $P^a P_a = 0$

Choose $p_a = (-E, 0, 0, E)$

Obtain $W_a = \lambda P_a$, where $\lambda \in \mathbb{R}$. (λ labels reps of $SO(2)$)

Topology (Global) { But rotation by angle θ changes wavefunction by $e^{i\theta\lambda}$. Lorentz not simply connected: Requires 4π rotation to be continuously connected to the identity.

$$\Rightarrow e^{i4\pi\lambda} = 1$$

$$\Rightarrow \lambda = \tfrac{1}{2}\mathbb{Z}, \quad \lambda = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \pm 2.$$

λ = helicity.

'neutrino' 'antineutrino' | photon gluon | graviton

$|\lambda|$ = "spin" of the particle

See Weinberg Vol I Chap 2

Coleman - Mandula Theorem

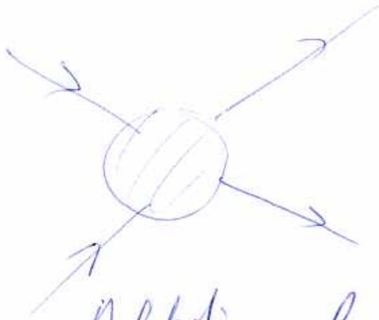
Phys. Rev. D 159 (1967) 1251

Follow E. Witten (1981) Erice lectures (see Weinberg Ch 24, App B). $D > 2$

In a theory with nontrivial scattering the only possible conserved quantities are P_a, M_{ab}, as well as "internal" symmetry charges Z_i which commute with P_a, M_{ab}. (electric charge, baryon # etc.)

"Proof" P_a and M_{ab} conservation leaves only the scattering angle unknown

$2 \to 2$ scattering.



2→2 scattering

Additional exotic conservation laws

=> scattering angle is determined up to a discrete set.

But the amplitude is always analytic

=> scattering angle would always be 0.

=> vanishes at all angles

<u>1+1 dims</u> theorem does not apply.

scattering angle $= 0, \pi$

<u>Example</u> Consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2$$

Theory has many conserved currents

$$J_\mu = \varphi_1 \partial_\mu \varphi_2 - \varphi_2 \partial_\mu \varphi_1 \qquad \partial^\mu J_\mu = 0$$

$$J_{\mu\nu} = \partial_\nu \varphi_1 \partial_\mu \varphi_2 - \varphi_2 \partial_\mu \partial_\nu \varphi_1 \qquad \partial^\mu J_{\mu\nu} = 0$$

$J_{\mu\nu\rho}$

no interaction that's why many cons. ~~law~~ quant.
(trivial scattering)

But for nontrivial scattering need interactions:

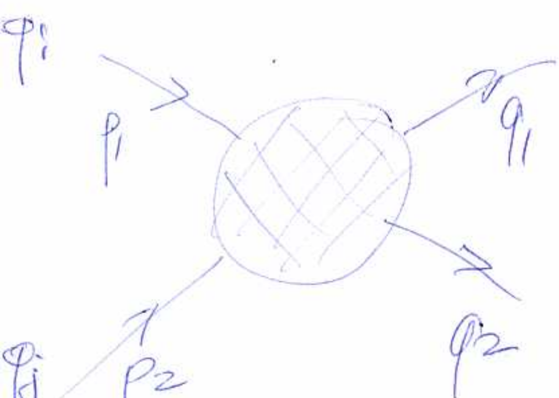
J_μ preserved $V(\varphi_1^2+\varphi_2^2)$.

not true of $J_{\mu\nu}, \ldots$

~~Suppose~~ Let $Q_{\rho\sigma} = \int d^{d-1}x\, J_{0\rho\sigma}$ be a conserved charge

Suppose $Q_{\rho\sigma}$ is symmetric, traceless

$$\langle p | Q_{\rho\sigma} | p \rangle \sim C(p^2)\left(p_\rho p_\sigma - \frac{1}{4}\eta_{\rho\sigma} p^2\right)$$



energy-mtm

$$p_{1\mu} + p_{2\mu} = q_{1\mu} + q_{2\mu}$$

$Q_{\rho\sigma}$ conservation $\Rightarrow$

$$p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} = q_{1\mu}q_{1\nu} + q_{2\mu}q_{2\nu}$$

$\Rightarrow$ $p_{1\mu} = q_{1\mu}$ or $p_{1\mu} = q_{2\mu}$

$\Rightarrow$ zero scattering angle

Consider adding a fermion to the theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{1}{2}\bar{\psi}_M \gamma^\mu \partial_\mu \psi_M$$

Majorana fermion

then

$$S_{\mu\alpha} = \partial_\rho(\phi_1 - i\phi_2)(\gamma^\rho \gamma_\mu \Psi_M)_\alpha$$ = fermionic current

with $\partial^\mu S_{\mu\alpha} = 0$

$$S_{n\alpha} = \partial_r(\phi_1 - i\phi_2)(\gamma^r \gamma_n \Psi_M)_\alpha$$

with $\partial^n S_{n\alpha} = 0$

Remarkably, $S_{\mu\alpha}$ remains conserved even if you add an interaction!

$$Q_\alpha = \int d^3x \, S_{0\alpha}$$ = conserved charge!

fermionic charge