

Phys 8911 Lecture 2 (M 24.01.2005)

Relation between Lorentz group and $SL(2, \mathbb{C})$

Consider matrices $\delta_m = (\delta_0, \delta_i)$ $\delta_i =$ Pauli matrices

Any real four-vector x^m can be used to construct a Hermitian X by

$$X = x^m \delta_m = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

where $X = X^\dagger$
and $\det X = -\eta_{mn} x^m x^n$

(Wess & Bagger App. A & B)

Under a transformation of the form

$$X \rightarrow X' = NXN^\dagger \quad \text{where } N \in SL(2, \mathbb{C})$$

where

$$\det X = \det X'$$

$$\eta_{mn} x^m x^n = \eta_{mn} x'^m x'^n \quad \text{since } \det N = 1$$

$X' = NXN^\dagger$ coincides with $x'^m = \Lambda^m_n x^n$
where $\Lambda \in SO(3, 1)$ (Lorentz transf.)

$SL(2, \mathbb{C})$ irreducible repⁿ

Denote $SL(2, \mathbb{C})$ indices by Greek letters

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$$N \in SL(2, \mathbb{C}) \rightarrow N_{\alpha}^{\beta} \quad \alpha, \beta = 1, 2.$$

$$N^* \rightarrow N_{\dot{\alpha}}^*{}^{\dot{\beta}}$$

Spinor representations.

$$(\frac{1}{2}, 0) \sim \Psi_{\alpha} \text{ transforms } \Psi'_{\alpha} = N_{\alpha}^{\beta} \Psi_{\beta}$$

$$(0, \frac{1}{2}) \sim \bar{\Psi}_{\dot{\alpha}} \text{ transforms } \bar{\Psi}'_{\dot{\alpha}} = N_{\dot{\alpha}}^*{}^{\dot{\beta}} \bar{\Psi}_{\dot{\beta}}$$

$$\uparrow \text{ "right-handed" } \equiv (\Psi_{\alpha})^*$$

Двухкомпонентный безреверсивный спинор
 • dot notation \leftrightarrow RH spinor

Introduce ϵ symbol. $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$ where

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$$

$$(\epsilon^{12} = 1, \epsilon_{12} = \epsilon^{21} = -1, \epsilon_{11} = \epsilon_{22} = 0)$$

(like $i\sigma_2$)

$\epsilon_{\alpha\beta} = N_{\alpha}^{\gamma} N_{\beta}^{\delta} \epsilon_{\gamma\delta} \rightarrow$ transforms as a tensor
of Lorentz group.

Can raise and lower indices with ϵ :

$$\Psi^{\alpha} = \epsilon^{\alpha\beta} \Psi_{\beta}; \quad \Psi_{\alpha} = \epsilon_{\alpha\beta} \Psi^{\beta}$$

Sim dotted indices.

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$$X' = N X N^T$$

$$\delta^m_{X'_m} = N \delta^m_{X_m} N^T =$$

$$= N_{\alpha}^{\dot{\beta}} \left(\delta^m_{\dot{\beta}} \right) X_m \left\{ N^{\dot{\alpha}} \right\} \dot{\beta}_2$$

δ^m must have index structure $\delta^m_{\dot{\alpha}\dot{\beta}} \sim \left(\frac{1}{2}, \frac{1}{2}\right)$ - vector representation.

Note Vectors and tensors (spinors) have Latin (greek) indices

Consider Ref. Ramond "Field Theory"
Weinberg Vol I Chap 2.

SL(2, C) transformation

$$N = 1 + \frac{i}{2} K_{ab} \delta^{ab}$$

where K_{ab} = infinitesimal parameter ($K_{ab} = -K_{ba}$)

$$\text{where } \delta^m_{\dot{\alpha}\dot{\beta}} = \frac{1}{4} (\delta^m_{\dot{\alpha}\dot{\beta}} - \delta^m_{\dot{\beta}\dot{\alpha}})$$

$$\text{where } \dot{\alpha}\dot{\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\gamma}\dot{\delta}} \delta^m_{\dot{\gamma}\dot{\delta}}$$

leads infinitesimal Lorentz transformation

$$T = 1 + \frac{i}{2} K_{ab} M^{ab}$$

where M^{ab} are the Lorentz generators satisfying

$$[M_{ab}, M_{cd}] = -i (\eta_{ad} M_{bc} - \eta_{ac} M_{bd} + \\ + \eta_{bc} M_{ad} - \eta_{bd} M_{ac}).$$

Lorentz
algebra

$$M^{ab} = -M^{ba}$$

Two and Four Component Spinors

Recall Dirac Spinors describe fermions
 $\frac{1}{2}$ integer spin particles \rightarrow 4 component
spinor field.

First, note that spinors are anticommuting
eg. $\psi_\alpha, \chi^\beta \rightarrow \{\psi_\alpha, \chi_\beta\} = \{\psi^\alpha, \bar{\chi}^\beta\} = \\ = \{\psi_\alpha, \bar{\chi}^\beta\} = 0$

Spinor summation convention: App. A & B
Wess & Bagger

$$\psi \chi \equiv \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha$$

$$\bar{\psi} \bar{\chi} \equiv \bar{\psi}_\alpha \bar{\chi}^\alpha$$

$$\psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \chi_\alpha = -\epsilon^{\beta\alpha} \psi_\beta \chi_\alpha = -\psi_\beta \chi^\beta = \\ = \chi^\alpha \psi_\alpha = \chi \psi$$

$$\bar{\Psi}\chi \equiv \bar{\Psi}_\alpha \chi^\alpha = -\bar{\Psi}^\alpha \chi_\alpha = \chi_\alpha \bar{\Psi}^\alpha = \bar{\chi}\Psi$$

To relate to Dirac spinors use Weyl repⁿ for γ -matrices.

$$\gamma^m = \begin{pmatrix} 0 & \delta^m \\ \bar{\delta}^m & 0 \end{pmatrix} \quad \begin{aligned} \bar{\delta}^0 &= \delta^0 = -\delta_0 \\ \bar{\delta}^i &= -\delta^i \end{aligned}$$

Clifford algebra $\{ \gamma^m, \gamma^n \} = 2\eta^{mn} \mathbb{1}$

Then a Dirac spinor is

$$\Psi_D = \begin{pmatrix} \Psi_\alpha \\ \bar{\chi}^\alpha \end{pmatrix} \begin{array}{l} \text{particle} \\ \text{antiparticle} \end{array} \quad \Psi_D = \Psi^\dagger \gamma_0 = \begin{pmatrix} \bar{\chi}^\alpha & \Psi_\alpha \end{pmatrix}$$

Dirac spinor combines 2-component spinors $\Psi_\alpha, \bar{\chi}^\alpha$

$$\text{Define } \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P_{L,R} = \frac{1}{2}(1 \pm \gamma^5)$$

$$P_L \Psi_D = \begin{pmatrix} \Psi_\alpha \\ 0 \end{pmatrix} \quad P_R \Psi_D = \begin{pmatrix} 0 \\ \bar{\chi}^\alpha \end{pmatrix}$$

Majorana spinor

$$\bar{\Psi}_M = \begin{pmatrix} \Psi_\alpha \\ \bar{\Psi}^\alpha \end{pmatrix} \text{ follows from requiring } \Psi = \chi$$