Phys 8911 Lecture 2 (M 24.01.2005)

Relation between Lorentz group and $SL(2,\mathbb{C})$

Consider matrices $\sigma_m = (\sigma_0, \sigma_i)$ $\quad \sigma_i$ = Pauli matrices

$\sigma_0 = \mathbb{1}$

Any real four-vector x^m can be used to construct a Hermitean X by

$$X = x^m \sigma_m = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

where $X = X^+$
and $\det X = -\eta_{mn} x^n x^m$

(Wess & Bagger App. A & B]

Under a transformation of the form

$$X \longrightarrow X' = N X N^+ \quad \text{where } N \in SL(2,\mathbb{C})$$

where
$\det X = \det X'$

$\eta_{mn} x^n x^m = \eta_{mn} x^{n\prime} x^{m\prime}$ since $\det N = 1$

$X' = N X N^+$ coincides with $x^{m\prime} = \Lambda^m{}_n x^n$
where $\Lambda \in SO(3,1)$ (Lorentz transf.)

$SL(2,\mathbb{C})$ irreducible rep^n

Denote $SL(2,\mathbb{C})$ indicies by Greek letters

$N \in SL(2,\mathbb{C}) \rightarrow N_\alpha{}^\beta \quad \alpha,\beta = 1,2.$

$N^* \longrightarrow N^*_{\dot\alpha}{}^{\dot\beta}$

Spinor representations.

$(\frac{1}{2},0) \sim \psi_\alpha$ transforms $\psi_\alpha' = N_\alpha{}^\beta \psi_\beta$

$(0,\frac{1}{2}) \sim \bar\psi_{\dot\alpha}$ transforms $\bar\psi_{\dot\alpha}' = N^*_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}$

"righthanded" $\equiv (\psi_\alpha)^*$

Двухкомпонентный Вейлевский спинор

- dot notation $\leftrightarrow$ RH spinor

Introduce ε symbol. $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$ where

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$$

$$(\varepsilon^{12} = 1,\ \varepsilon_{12} = \varepsilon^{21} = -1,\ \varepsilon_{11} = \varepsilon_{22} = 0)$$

(like $i\sigma_2$)

$\varepsilon_{\alpha\beta} = N_\alpha{}^\gamma N_\beta{}^\delta \varepsilon_{\gamma\delta} \rightarrow$ transfforms as a tensor of Lorentz group.

Can raise and lower indices with ε:

$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$; $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$

Sim dotted indices.

2

$X' = N X N^{+}$

$\sigma^m x'_m = N \sigma^m x_m N^{+} =$

$= N_\alpha{}^\beta (\sigma^m)_{\beta\dot\beta}\, x_m \, \{N^*\}^{\dot\beta}{}_{\dot\alpha}$

σ^m must have index structure $\sigma^m_{\alpha\dot\alpha} \sim \left(\frac{1}{2},\frac{1}{2}\right)$ – vector representation.

<u>Note</u> Vectors and tensors (spinors) have Latin (greek) indices

Consider <u>Ref.</u> Ramond "Field theory"
Weinberg Vol I Chap 2.

$SL(2,\mathbb{C})$ transformation

$N = 1 + \frac{i}{2} k_{ab} \sigma^{ab}$

where k_{ab} = infinitesimal parameter ($k_{ab} = -k_{ba}$)

where $\sigma^{mn} = \frac{1}{4}\left(\sigma^n \bar\sigma^m - \sigma^m \bar\sigma^n\right)$

where $\bar\sigma^{m\dot\alpha\alpha} = \varepsilon^{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \sigma^m_{\beta\dot\beta}$

leads infinitesimal Lorenz transformation

$T = \mathbb{1} + \frac{i}{2} k_{ab} M^{ab}$

where M^{ab} are the Lorentz generators statisfying

$$[M_{ab}, M_{cd}] = -i\left(\eta_{ad} M_{bc} - \eta_{ac} M_{bd} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac}\right).$$

Lorenz algebra $M^{ab} = -M^{ba}$

Two and Four Component Spinors

Recall Dirac Spinors describe fermions $\frac{1}{2}$ integer spin particles $\leftrightarrow$ 4 component spinor field.

First, note that spinors are anticommuting

eg. $\psi_\alpha, \bar{\chi}^{\dot\alpha} \rightarrow \{\psi_\alpha, \chi_\beta\} = \{\bar\psi^{\dot\alpha}, \bar\chi^{\dot\beta}\} = \{\psi_\alpha, \bar\chi^{\dot\beta}\} = 0$

Spinor summation convention: App. A & B Wess & Bagger

$$\psi\chi \equiv \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha} \bar\chi^{\dot\alpha}$$

$$\psi^\alpha \chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta \chi_\alpha = -\varepsilon^{\beta\alpha}\psi_\beta\chi_\alpha = -\psi_\beta \chi^\beta = \chi^\alpha \psi_\alpha = \chi\psi$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi$$

To relate to Dirac spinors use Weyl repⁿ for γ-matrices.

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix} \qquad \begin{aligned} \bar\sigma^0 &= \sigma^0 = -\sigma_0 \\ \bar\sigma^i &= -\sigma^i \end{aligned}$$

Clifford algebra $\{\gamma^m \gamma^n\} = 2\eta^{mn}\mathbb{1}$

Then a Dirac spinor is

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \begin{array}{l} \leftarrow \text{particle} \\ \leftarrow \text{antiparticle} \end{array} \qquad \bar\Psi_D = \Psi^\dagger\gamma_0 = (\chi^\alpha, \bar\psi_{\dot\alpha})$$

Dirac spinor combines 2-comp spinors $\psi_\alpha, \bar\chi^{\dot\alpha}$

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)$$

$$P_L \Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \qquad P_R \Psi_D = \begin{pmatrix} 0 \\ \bar\chi^{\dot\alpha} \end{pmatrix}$$

<u>Majorana spinor</u>

$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$ follows from requiring $\psi = \chi$