Math 8306 Lecture 9 (M 09.27.2004)

Singular Homology

$S_n(X) := \{T : \Delta^n \to X\} \rightsquigarrow C_n(X) = \langle S_n(X) \rangle_{\mathbb{Z}}$

set

group of singular chains

(free abelian group, generated by this set)

$C_n(X;G) := C_n(X) \otimes_{\mathbb{Z}} G$

$d = \sum_{i=0}^{n} (-1)^i d_i : C_n(X;G) \to C_{n-1}(X,G)$

$d^2 = 0$

$H_n(X;G) := H_n(C_\bullet(X;G), d)$

inclusion respects differentials

(X,A) topological pair : $0 \to C_\bullet(A) \to C_\bullet(X) \to$

$\to C_\bullet(X,A) \to 0$

quotient complex

$C_\bullet(X)/C_\bullet(A)$

short exact sequence

$0 \longrightarrow C_\bullet(A;G) \to C_\bullet(X;G) \to C_\bullet(X,A;G) \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad C_\bullet(X;G)/C_\bullet(A;G)$

Homology of $(X,A;G)$:

$H_n(X,A;G) := H_n(C_\bullet(X,A;G), d)$

$\rightsquigarrow \partial : H_n(X,A;G) \to H_{n-1}(A;G)$ by Snake Lemma and a LES

$$\dots \longrightarrow H_n(A;G) \longrightarrow H_n(X;G) \longrightarrow H_n(X,A;G) \xrightarrow{\partial} H_{n-1}(A;G) \to$$

Checking most axioms is trivial

<u>Functoriality</u>: $(X,A) \xrightarrow{f} (Y,B)$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \cup \\ A & \xrightarrow{f|_A} & B \end{array}$$

$f_*(T) = f \circ T$

$\Delta^n \xrightarrow{T} X \xrightarrow{f} Y$

$$\begin{array}{ccc} C_\bullet(X,G) & \xrightarrow{f_*} & C_\bullet(Y;G) \\ \uparrow & & \uparrow \\ C_\bullet(A;G) & \xrightarrow{(f|_A)_*} & C_\bullet(B;G) \end{array}$$

$\therefore$ quotients

$$C_\bullet(X,A;G) \xrightarrow{f_*} C_\bullet(Y,B;G) \overset{H_\bullet}{\rightsquigarrow}$$

$$H_\bullet(X,A;G) \xrightarrow{f_*} H_\bullet(Y,B;G)$$

$id_+ = id$

$(fg)_+ = f_+ g_+$

$A^n \xrightarrow{T} X \xrightarrow{g} Y \xrightarrow{f} Z$

$(f \circ g)_*(T) = (f \circ g) \circ T$

$f_*(g_*(T)) = f \circ (g \circ T)$

<u>Naturality of ∂</u> : $(X,A) \xrightarrow{f} (Y,B)$

$$\begin{array}{ccc} H_n(X,A;G) & \xrightarrow{\partial} & H_{n-1}(A;G) \\ f_* \downarrow & \circlearrowright & \downarrow f_* \\ H_n(Y,B,G) & \xrightarrow{\partial} & H_{n-1}(B,G) \end{array}$$

g

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(A;G) \arrow[r] \arrow[d, "f_*"] & C_\bullet(X;G) \arrow[r] \arrow[d, "f_*"] & C_\bullet(X,A;G) \arrow[r] \arrow[d, "f_*"] & 0 \\ 0 \arrow[r] & C_\bullet(B;G) \arrow[r] & C_\bullet(Y;G) \arrow[r] & C_\bullet(Y,B;G) \arrow[r] & 0 \end{tikzcd}$$

Enhanced Snake Lemma ~~involves naturality~~.

Dimension: $H_q(*;G) = \begin{cases} G & q=0 \\ 0 & q\neq 0 \end{cases}$

$\exists!\ T : \Delta^n \to \{*\}$

$C_n(*;G) = G \quad \forall n \geq 0.$

$C_n(*;G) = 0 \quad \forall n < 0.$

$C_n(*) = \mathbb{Z}$

$\mathbb{Z} \otimes G = G$

$1 \otimes g \leftarrow g$

$\sum m_i g_i = \sum 1 \otimes m_i g$

$\downarrow$

$\sum m_i g_i \in G$

$R \otimes_R M = M$

$d_i(T_n) = T_{n-1} \quad \forall i \quad n \geq 1$

$d_i(T_0) = 0$

$d(T_n) = \sum_{i=0}^{n} d_i(T_n)(-1)^i = \sum_{i=0}^{n} (-1)^i T_{n-1} = \begin{cases} T_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

$d : C_n \to C_{n-1}$

$\overset{\shortparallel}{\sum d_i (-1)^i}$

$d(T_0) = 0$

$$\cdots \xrightarrow{id} G \xrightarrow{0} G \xrightarrow{id} G \xrightarrow{0} G \xrightarrow{d} \cdots \longrightarrow G \xrightarrow{id} G \xrightarrow{0}$$

$GT_n \quad GT_{n-1} \quad GT_{n-2} \qquad GT_1 \quad GT_0.$

$$H_n(*;G) = \begin{cases} 0/0 = 0 & n \text{ even} \\ G/G = 0 & n \text{ odd} \end{cases} \quad n \geq 1$$

<u>Homotopy axiom</u>: $f, g : (X,A) \to (Y,B)$, $f \simeq g$
$\Rightarrow f_* = g_*$

<u>Excision Axiom</u>: A, B subcomplexes of CW cx X
$$H_n(A, A \cap B, G) \xrightarrow{\sim} H_n(X, B; G).$$