

Homology of spheres

$\tilde{H}_p(\mathbb{D}^n; G) = 0$, because $\mathbb{D}^n \sim \underline{G} * \{*\}$

$H_p(X, A; G) = H_p(X, A) \rightsquigarrow \tilde{H}_p(X; G) := H_p(X, *)$
 \parallel
 $\tilde{H}_p(X)$

Prop. $\tilde{H}_p(S^0) = \begin{cases} G & p=0, \\ 0 & p \neq 0 \end{cases}$

Proof: $H_p(S^0) = H_p(+)\oplus H_p(+)$ $\approx \begin{cases} G \oplus G & p=0 \\ 0 & p \neq 0 \end{cases}$
 $S^0 = \{-1, 1\} \subset \mathbb{R}$ $* := -1 \in S.$

$\xrightarrow{0} H_p(*) \xrightarrow{\cong} H_p(S^0) \rightarrow H_p(S^0, *) \xrightarrow{0}$
 $p=0 \xrightarrow{f} G \rightarrow G \oplus G$
 $* \mapsto -1$ \parallel
 $\tilde{H}_p(S^0)$

$\tilde{H}_p(S^0) = \begin{cases} G & p=0 \\ 0 & p \neq 0 \end{cases}$ \square

Prop $\tilde{H}_p(S^n) = \begin{cases} G & p=n \\ 0 & p \neq n, \end{cases}$

Prop. $\tilde{H}_p(S^n) \cong D^n, \partial D^n = S^{n-1}$

$$S^n \cong D^n / \partial D^n$$

$$(D^n, \partial D^n)$$

$$\dots \rightarrow \tilde{H}_p(S^{n-1}) \rightarrow \tilde{H}_p(D^n) \rightarrow \tilde{H}_p(D^n / \partial D^n) \cong \tilde{H}_p(S^n)$$

$$\Rightarrow \tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1}) = \begin{cases} \mathbb{Q} & p=n \text{ by induction} \\ 0 & p \neq n \end{cases}$$

Prop. $S^n \xrightarrow{id} S^n$

$n \geq 0$

$f: S^n \rightarrow * \in S^n$ (i.e. $f: S^n \rightarrow S^0, f(x) = * \in S^0, \forall x \in S^n$)

Then $f \neq id$

Proof: $id_*: \tilde{H}_n(S^n) \rightarrow \tilde{H}_n(S^n)$

$\cong id_{\mathbb{Q}}$ (because homology is a functor)

$f_* \neq id$

$f: S^n \rightarrow \{*\} \rightarrow S^n$

$f_*: \tilde{H}_n(S^n) \rightarrow \tilde{H}_n(*) \rightarrow \tilde{H}_n(S^n)$

$\cong 0 \Rightarrow f_* = 0 \neq id_{\mathbb{Q}}$, unless $\mathbb{Q} = \{0\}$
(contractible space)
reduced homology of the point is 0.

Singular Homology

A way to construct a homology theory

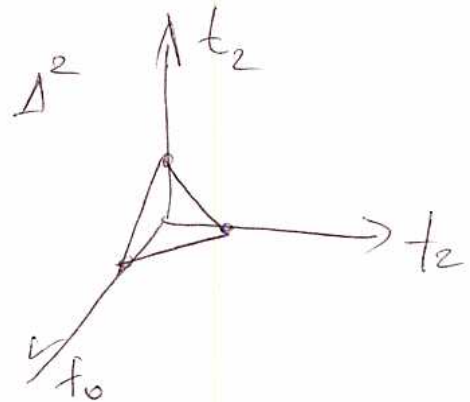
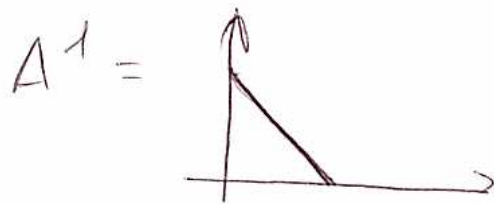
$H_*(X, A; G)$ for any abelian group G .

gen for (X, A) a topological pair ($A \subset X, X$ top. space)

First let's define $H_*(X; G)$ for an arbitrary top. space X .

n -Simplex $\Delta^n := \{ (t_0, t_1, \dots, t_n) \mid 0 \leq t_i \leq 1 \ \forall i, \sum_{i=0}^n t_i = 1 \}$
 $\subset \mathbb{R}^{n+1}$

$$\Delta^0 = \{ \pm 1 \} \subset \mathbb{R}$$



Δ^3 tetrahedron

$$\Delta \subset \mathbb{R}^4$$

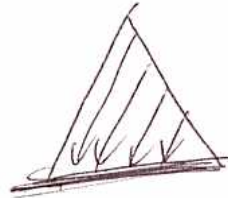
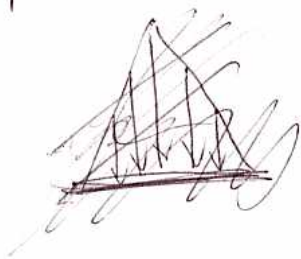
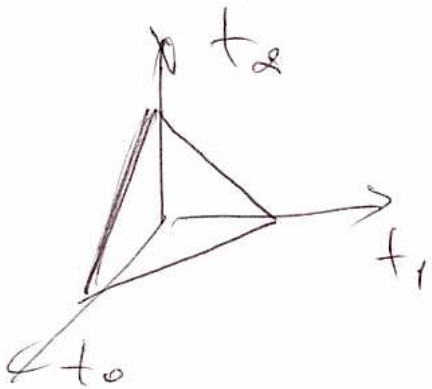
face maps: $\partial_i: \Delta^{n-1} \rightarrow \Delta^n \quad i=0, 1, \dots, n$

$$\partial_i (t_0, t_1, \dots, t_{n-1}) := (t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

degeneracy maps: $\sigma_i: \Delta^{n+1} \rightarrow \Delta^n, \quad i=0, 1, \dots, n$

$$\sigma_i(t_0, t_1, \dots, t_{n+1}) := (t_0, t_1, \dots, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1})$$

$$\sigma_0(t_0, t_1, t_2) = (t_0 + t_1, t_2)$$



X top. space

$$S_n X = \{f: \Delta^n \rightarrow X \mid f \text{ continuous}\}$$

= singular n -simplices ~~for~~ of X .

face operator $d_i: S_n X \rightarrow S_{n-1} X, i=0, \dots, n$.

$$(d_i f)(t) = f(d_i(t))$$

$$f: \Delta^n \rightarrow X$$

$$d_i f: \Delta^{n-1} \rightarrow X$$

degeneracy operator $s_i: S_n X \rightarrow S_{n+1} X,$

$$(s_i f)(t) = f(\sigma_i(t)) \quad t \in \Delta^{n+1} \quad i=0, \dots, n.$$

Lemma: $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$

$$\text{Define } C_0(X, G) := C_0(X) \otimes G$$

$C_n(X) := \langle S_n X \rangle$ free abelian group generated by the set of n -simplices

$$d : C_n(X) \rightarrow C_{n-1}(X)$$

$$d := \sum_{i=0}^n (-1)^i d_i$$

Lemma $d^2 = 0$ (follows from Lemma above)

$$\overline{d} := d \otimes \text{id}_G = C_n(X; G) \rightarrow C_{n-1}(X; G)$$

$$H_n(X; G) := H_n(C_*(X; G), \overline{d})_{n \in \mathbb{Z}} \text{ (of form)}$$

Singular Homology. $\text{Ker } \overline{d} / \text{Im } \overline{d}$.

for CW pairs satisfies the axioms. ~~to f~~

$$H_* = \bigoplus_{n \in \mathbb{Z}} H_n$$