Math 8306 (F 09.24.04) Lecture 8

Homology of spheres

$\tilde{H}_p(D^n;G) = 0$, because $D^n \sim \{*\}$

$H_p(X,A;G) = H_p(X,A) \rightsquigarrow \tilde{H}_p(X;G) := H_p(X,*)$

$$\tilde{H}_p(X;G) = \tilde{H}_p(X)$$

Prop. $\tilde{H}_p(S^0) = \begin{cases} G & p=0, \\ 0 & p\neq 0 \end{cases}$

Proof: $H_p(S^0) = H_p(+) \oplus H_p(+) \cong \begin{cases} G\oplus G & p=0 \\ 0 & p\neq 0 \end{cases}$

$S^0 = \{-1,1\} \subset \mathbb{R} \qquad * := -1 \in S.$

$$\xrightarrow{0} H_p(*) \longrightarrow H_p(S^0) \to H_p(S^0,*) \xrightarrow{0}$$

$$p=0 \quad \xrightarrow{\cancel{p}} \quad G \longrightarrow G\oplus G$$

$$* \longmapsto -1$$

$$H_p(S^0,*) = \tilde{H}_p(S^0)$$

$\tilde{H}_p(S^0) = \begin{cases} G & p=0 \\ 0 & p\neq 0 \end{cases}$ ■

Prop $\tilde{H}_p(S^n) = \begin{cases} G & p=n \\ 0 & p\neq n, \end{cases}$

<u>Proof</u> ~~$\tilde{H}_p(S^n)$~~ $D^n, \partial D^n = S^{n-1}$

$S^n \cong D^n / \partial D^n$

$(D^n, \partial D^n)$

$$\cdots \to \tilde{H}_p(S^{n-1}) \to \underset{0}{\tilde{H}_p(D^n)} \to \tilde{H}_p(\underbrace{D^n/\partial D^n}_{= S^n}) \to \tilde{H}_{p-1}(S^{n-1}) \to \cdots$$

$$\Rightarrow \tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1}) = \begin{cases} G & p = n \text{ by induction} \\ 0 & p \neq n \end{cases}$$

∎

<u>Prop.</u> $S^n \xrightarrow{id} S^n$ $n \geq 0$

$f : S^n \to * \in S^n$ (i.e. $f : S^n \to S^n$, $f(x) = * \in S^n$ $\forall x \in S^n$).

Then $f \not\simeq id$

<u>Proof</u>: $id_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$

$\quad = id_G$ (because homology is a functor)

$f_* \overset{?}{\neq} id$

$f : S^n \to \{*\} \to S^n$

$f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(*) \to \tilde{H}_n(S^n)$

$\tilde{H}_n(*) = 0$ (contractable space) $\Rightarrow f_* = 0 \neq id_G$, unless $G = \{0\}$

reduced homology of the point is $\varnothing$.

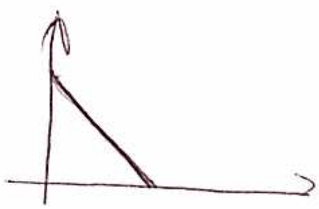
∎

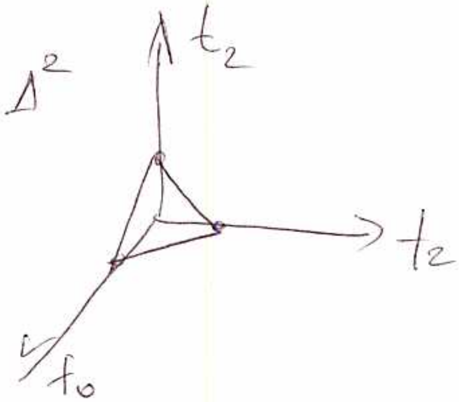
Singular Homology

A way to construct a homology theory $H_\cdot(X,A;G)$ for any abelian group G.
gen for (X,A) a topological pair ($A\subset X$, X top. space)
First let's define $H_\cdot(X;G)$ for an arbitrary top. space X.

Simplex $\Delta^n := \{(t_0,t_1,\dots,t_n) \mid 0\le t_i\le 1\ \forall i, \sum_{i=0}^{n} t_i = 1\} \subset \mathbb{R}^{n+1}$
n-Simplex

$\Delta^0 = \{1\} \subset \mathbb{R}$

$\Delta^1 =$ 

Δ^2 

Δ^3 tetrahydron

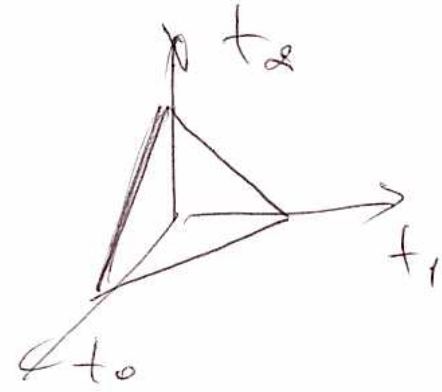
 $\subset \mathbb{R}^4$

face maps $\delta_i : \Delta^{n-1} \to \Delta^n \quad i=0,1,\dots,n$

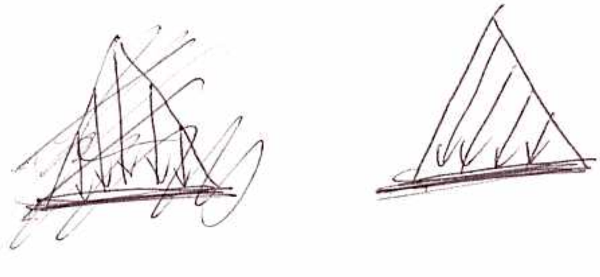
$\delta_i(t_0,t_1,\dots,t_{n-1}) := (t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$

degeneracy maps: $\sigma_i : \Delta^{n+1} \to \Delta^n$, $i=0,1,\dots,n$

$\sigma_i(t_0, t_1, \dots, t_{n+1}) := (t_0, t_1, \dots, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1})$



$\sigma_0(t_0, t_1, t_2) = (t_0 + t_1, t_2)$.



X top. space

$S_n X = \{ f : \Delta^n \to X \mid f \text{ continuous} \}$

$=$ singular n-simplices ~~for~~ of X.

face operator $d_i : S_n X \to S_{n-1} X$, $i = 0, \dots, n$.

$(d_i f)(t) = f(\partial_i(t))$

$f : \Delta^n \to X$

$d_i f : \Delta^{n-1} \to X$

degeneracy operator $S_i : S_n X \to S_{n+1} X$, $i = 0, \dots, n$.

$(s_i f)(t) = f(\sigma_i(t)) \quad t \in \Delta^{n+1}$

<u>Lemma</u>: $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$

Define $C_\cdot(X; G) := C_\cdot(X) \otimes G$

$C_n(X) := \langle S_n X \rangle$ free abelian group generated by the set of n-simp

$d : C_n(X) \to C_{n-1}(X)^{\delta}$

$d := \sum_{i=0}^{n} (-1)^i d_i$

Lemma $d^2 = 0$ (follows from Lemma above)

$d := d \otimes id_G = C_n(X;G) \to C_{n-1}(X;G)$

$H_n(X;G) := H_n(C_\cdot(X;G), d)_{n\in\mathbb{Z}}$ (oforms

Singular Homology. $\operatorname{Ker} d / \operatorname{Im} d$.

for CW pairs satisfies the axioms. ~~of~~

$H_\cdot = \bigoplus_{n\in\mathbb{Z}} H_n$