Generalized Homology

(drop Dimension Axiom)

$E_q(X,A)$

Reduced vs. Unreduced Homology

X based CW complex

$\tilde{E}_q(X) := E_q(X, *)$

reduced homology; star; base point

Proposition: $\tilde{E}_q(X) \oplus E_q(*) = E_q(X)$ And $\tilde{E}_q(X) : \operatorname{Ker}(E_q(X) \to E_q(*))$

Proof: LES

$$\cdots \xrightarrow{\partial} E_q(*) \underset{r}{\overset{i}{\rightleftarrows}} E_q(X) \to \underbrace{E_q(X,*)}_{\tilde{E}_q(X)} \to E_{q-1}(*) \xrightarrow{\partial} \cdots$$

natural map $r = E_q(X \to *)$

$ri = id$ $\qquad i\partial = 0 \overset{r}{\Rightarrow} \partial = 0$

$\Downarrow$ Algebra

$\partial = 0$

$$\Rightarrow E_q(X) \cong E_q(*) \oplus \underbrace{E_q(X,*)}_{\tilde{E}_q(X)} \quad \blacksquare$$

$* \in A \subset X$ based subcomplex

basepoint for the subcomplex is a basepoint for the complex.

$E_q(*)$ maps isomorphically under $E_q(A)\to E_q(X)$

$\Rightarrow \tilde{E}_q(A)\to \tilde{E}_q(X)$

$\Rightarrow$ LES for reduced homology

$$\cdots \to \tilde{E}_q(A) \to \tilde{E}_q(X) \to \tilde{E}_q(X,A) \to \tilde{E}_{q-1}(A) \to \cdots$$
$$\xrightarrow{0} E_q(*) \xrightarrow{\cong} E_q(*) \xrightarrow{0} 0 \xrightarrow{0} E_{q-1}(*) \xrightarrow{\cong} E_{q-1}(*)$$

Excision axiom for unreduced hom $\to$ Suspension axiom for reduced

From reduced to unreduced

X (unbased) CW complex $\rightsquigarrow X_+ = X \sqcup \{*\}$ based CW complex
add an extra point (base point)

$E_q(X_+) = E_q(X) \oplus E_q(*)$ want this to be true

$E_q(X) := \tilde{E}_q(X_+)$

$E_q(*) = \tilde{E}_q(* \sqcup *)$ "coefficients"

$X \to Y$ cellular map $\rightsquigarrow X_+ \to Y_+$

$f_*:$ ~~E_q~~ $\tilde{E}_0(X_+) \to \tilde{E}_0(Y_+)$

$f_*: E_0(X) \to E_0(X)$

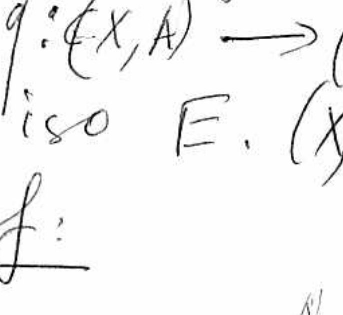
Theorem: If $i : A \hookrightarrow X$ subcomplex, then $q : (X,A) \longrightarrow (X/A, *)$ induces iso $E_\cdot(X,A) \to E_\cdot(X/A, *) = \tilde{E}_\cdot(X/A)$

Proof:

unreduced cone

$C_i = X \cup_i CA$

where $CA = A \times I / A \times \{1\}$



Choose a CW structure on C_i

C_i union of its subcomplexes $X \cup_i A \times [0, 2/3]$ and $A \times [\frac{1}{3}, 1] / (A \times \{1\})$

$(A, A \cap B) \hookrightarrow (X, B)$

$$
\begin{tikzcd}
(X \cup_i A \times [0, 2/3], A \times [\frac{1}{3}, \frac{2}{3}]) \arrow[r, hook, "\text{iso on } E_\cdot"] \arrow[d, "\text{homotopy equivalence}"'] & (C_i, A \times [\frac{1}{3}, 1] / A \times \{1\}) \arrow[d, "\text{quotient by } CA"] \\
(X, A) \arrow[r, "q"', "\text{quotient map}"] & (X/A, *)
\end{tikzcd}
$$

Apply $E_\cdot$ to the comm. diagram

$\Rightarrow E_\cdot(q)$ is an isomorphism

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LES for reduced homology

$$\cdots \to \tilde{E}_q(A) \to \tilde{E}_q(X) \to \tilde{E}_q(X/A) = \tilde{E}_{q-1}(A) \to \cdots$$