

Chain Maps and Homotopies

$$f: X \rightarrow Y, \quad \begin{array}{c} \overrightarrow{\square} \\ \downarrow \text{G} \\ \overrightarrow{\square} \end{array} \Rightarrow H \# f_2 = H \cdot (f): H_*(X) \rightarrow H_*(Y)$$

$$s: X \rightarrow Y_{+1} \quad ds + sd = f - g$$

same maps of Homology

Tensor Product

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$$

$$d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$$

$$d_{X \otimes Y} = d_X \otimes id + id \otimes d_Y$$

$$\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \quad 0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \rightarrow 0$$

$$[1] \quad [0] \quad [1]$$

$$d[1] = [1] - [0]$$

Then a homotopy ^{between maps} is $X \otimes I \xrightarrow{h} Y$, s.t. $s[1] = f, s[0] = g$

$$h(x \otimes [1]) = f(x)$$

$$h(x \otimes [0]) = g(x)$$

$$h(x \otimes [1]) = (-1)^{|x|} s(x) \quad \#$$

Property: Consequences of Axioms

$E_q(X, A)$ generalized in theory (when dimension axiom is dropped)

Existence and Uniqueness if you have $E_*(X, A) \rightarrow E'_*(X, A)$ naturally iso on $(X, A) = \mathbb{K}^n, \mathbb{R}^n$

$$\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$$

Reduced, based vs unbased

$\bar{E}_q(X) = E_q(X, *)$, if X is based CW complex

X retracts on $*$ $\Rightarrow E_*(X) \cong \bar{E}_*(X) \oplus E_*(*)$ and $\bar{E}_*(X) = \text{Ker}(E_*(X) \rightarrow E_*(*))$

$$\text{because } \rightarrow E_q(*) \rightarrow E_q(X) \rightarrow \bar{E}_q(X) \rightarrow E_{q-1}(X)$$

$v \in A \subset X$ $E_*(v)$ maps isomorphically under $E_*(A) \rightarrow E_*(X)$

$$\Rightarrow LES \quad \dots \rightarrow E_q(A) \rightarrow \bar{E}_q(X) \rightarrow E_q(X, A) \rightarrow \bar{E}_{q-1}(A) \rightarrow \dots$$