Math 8306 Lecture 4 (W 09.15.2004)

Axiomatic Homology Theory

G abelian, (X,A) CW pairs

Theorem: $\forall q \in \mathbb{Z}$ $\exists$ functors $H_q(X,A;G)$ from the homotopy cat. of CW pairs (X,A) to the cat. of ab. groups and natural transformations $\partial : H_q(X,A;G) \to H_{q-1}(A;G)$ satisfying and being determined by the following axioms:

Dimension $X = \{*\}$, $H_q(X,G) = \begin{cases} G & q=0 \\ 0 & q \neq 0 \end{cases}$

Exactness: the sequence

$$\cdots \to H_q(A;G) \to H_q(X;G) \to H_q(X,A;G) \xrightarrow{\partial} H_{q-1}(A;G) \to \cdots$$

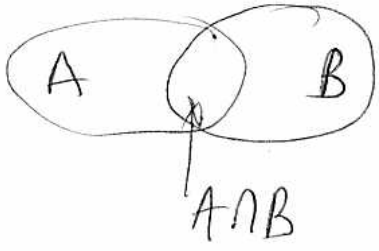
is exact

Excision: If $X = A \cup B$, A, B subcomplexes of X, then $(A, A\cap B) \hookrightarrow (X,B)$ induces an isomorphism

$$H_q(A, A\cap B; G) \xrightarrow{\simeq} H_q(X,B;G)$$

homology of X relative to B.

X:



Additivity: $(X,A) = \coprod_{n\in J} (X_n, A_n)$

Then the inclusions $(X_n, A_n) \hookrightarrow (X, A)$ induce an isomorphism

$$\bigoplus_n H_q(X_n, A_n; G) \to H_q(X, A; G), \text{ for each } q\in\mathbb{Z}$$

Exercise: When summation is finite the additivity axiom follows from the other axioms.

$$H_\bullet(X, A; G) := \bigoplus_{q\in\mathbb{Z}} H_q(X, A; G). \quad (\text{graded abelian group})$$

Remarks about categories:

Category $\mathcal{C}$ is a collections $Ob\,\mathcal{C}$ of objects and morphisms $Mor(A,B)$ for $A,B\in Ob\,\mathcal{C}$ satisfying certain natural properties

E.g.: (X,A) CW pairs form a category

$Ob(\text{CW-pairs}) = \{\text{CW pairs } (X,A)\}$

$Mor((X,A),(Y,B)) = \{\text{cellular maps } (X,A)\to(Y,B)\}$

$Ob(h\text{-CW-pairs}) = \{\text{cw pairs } (X,A)\}$

homotopy

$Mor_h((X,A),(Y,B)) = \{\text{c.h. classes of cell maps } (X,A)\to(Y,B)\}$

Functor $H: \mathcal{C}\to\mathcal{D}$

$H(C)\in Ob\,\mathcal{D} \quad \forall C\in Ob\,\mathcal{C}$

$$C \xrightarrow{f} C' \rightsquigarrow^{H} H(C) \xrightarrow{H(f)} H(C')$$

$$\text{Mor}_{\mathcal{C}}(C, C')$$

<u>Natural transformation</u>: a homotopy between functors

H_1, H_2 functors $\mathcal{C} \to \mathcal{D}$ ~~a cat~~

a natural transformation ∂ from H_1 to H_2 is a collection of morphisms in $\mathcal{D}$

$\forall\ C \in \text{Ob}\,\mathcal{C}$ satisfying $f: C \to C'$

$$H_1(C) \xrightarrow{\partial} H_2(C)$$

$$\begin{array}{ccc} H_1(C) & \xrightarrow{\partial} & H_2(C) \\ {\scriptstyle H_1(f)}\downarrow & \circlearrowright & \downarrow{\scriptstyle H_2(f)} \\ H_1(C') & \xrightarrow{\partial} & H_2(C') \end{array}$$

коммутативная диаграмма

In the theorem

$$\partial: H_q(X, A; G) \to H_{q-1}(A; G)$$

thought of as a functor

$$F_{q-1}(X, A; G) = H_{q-1}(A; G).$$

<u>Homological algebra</u>

R commutative case

M an R-module

(e.g. $R = \mathbb{Z}$, a $\mathbb{Z}$-module same as an ab. group.

a (chain) complex $X_\bullet$ is

$$\cdots \to X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \to \cdots$$

(R-modules and homomorphisms)

satisfying $d_i \circ d_{i+1} = 0$

$\Updownarrow$

$(\ker d_i \supset \operatorname{Im} d_{i+1})$

<u>cochain complex $X^\bullet$:</u>

$$\cdots \to X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \to \cdots$$

$d^i d^{i-1} = 0$

(or $d^2 = 0$)

$H_i(X_\bullet) = \ker d_i / \operatorname{Im} d_{i+1}$

$H_\bullet(X_\bullet) = \bigoplus_{i \in \mathbb{Z}} H_i(X_\bullet)$

same for ~~co~~ cochain complexes gives cohomology groups

<u>Exact sequence</u>: a complex with zero homology (i.e., $\operatorname{Im} d_{i+1} = \ker d_i \ \forall i$) = ~~acyc~~ acyclic complex.

$0 \to X \to Y \to Z \to 0$ is short exact sequence

<u>Theorem</u> : A short exact sequence of complexes

$$0 \to X_\bullet \to Y_\bullet \to Z_\bullet \to 0.$$

(i.e.

$$\begin{tikzcd}
0 \arrow[r] & X_{i+1} \arrow[r] \arrow[d, "d"] \arrow[dr, phantom, "\circlearrowleft"] & Y_{i+1} \arrow[r] \arrow[d, "d"] \arrow[dr, phantom, "\circlearrowleft"] & Z_{i+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & X_i \arrow[r] \arrow[d, "d"] \arrow[dr, phantom, "\circlearrowleft"] & Y_i \arrow[r] \arrow[d, "d"] \arrow[dr, phantom, "\circlearrowleft"] & Z_i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & X_{i-1} \arrow[r] \arrow[d] & Y_{i-1} \arrow[r] \arrow[d] & Z_{i-1} \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

with exact rows) $\Rightarrow$ ~~long e~~

<u>long exact sequence</u>

$$\cdots \to H_i(X_\bullet) \to H_i(Y_\bullet) \to H_i(Z_\bullet) \xrightarrow[\partial]{} H_{i-1}(X_\bullet)$$

diagram chase method. connecting homomorphism

HW: Proove the theorem