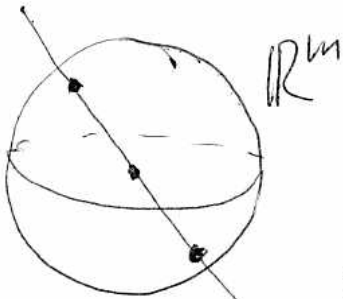


CW Complexes

More examples:

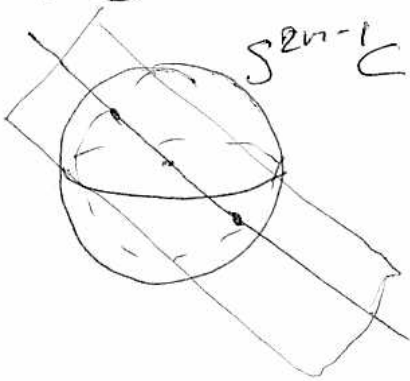
1. $\mathbb{R}P^n = \{ \text{lines in } \mathbb{R}^{n+1} \}$
 contains one m -cell for each $m \leq n$,
 attached via

$$\partial D^m = S^{m-1} \rightarrow \mathbb{R}P^{m-1} \quad (\text{standard double cover})$$



2. $\mathbb{C}P^n = \{ \text{complex lines in } \mathbb{C}^{n+1} \}$
 contains one $2m$ -cell for each
 $m: 2m \leq n$, attached via

$$\partial D^{2m} = S^{2m-1} \rightarrow \mathbb{C}P^{m-1}$$



$(2m-1)$ -skeleton

fiber bundle

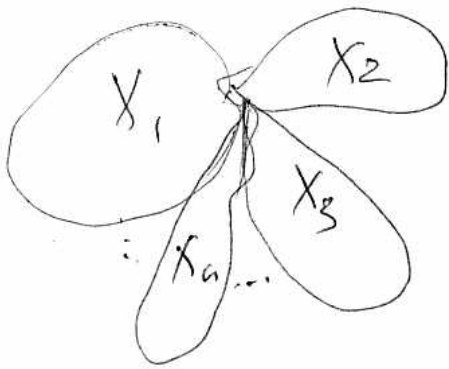
fiber - circle

reference number of \mathbb{C}^m
 cofibration

" fiber bundle with fiber S^1 "

Operations with CW complexes

1. $\bigvee_i X_i = \bigsqcup_i X_i / x_i^0 \sim x_j^0$, where x_i^0 is a basepoint chosen at a vertex of X_i (which is a CW complex)
- ↑
(disjoint union)



wedge = bouquet
has one m -cell for each cell of X_i and each i ($m \geq 1$)

2. X, Y CW complexes, $A \subset X$ subcomplex
 $f: A \rightarrow Y$ cellular (preserves the skeleton)

Then $Y \cup_f X$ is a CW complex containing Y as a subcomplex and one cell for each cell of X not in A .

$(Y \cup_f X) / Y \cong X/A$ (isomorphism of CW complexes)

3. Colimit (here: union) (direct limit, inductive limit, inverse limit)

$X_n \subset X_{n+1}$ (inclusion as a subcomplex)

Then $\bigcup_n X_n$ is a CW complex, containing X_n as a subcomplex.

4. X, Y CW complexes

$X \times Y$ a CW complex with an n -cell for each pair (p -cell of X, q -cell of Y) for $p+q=n$

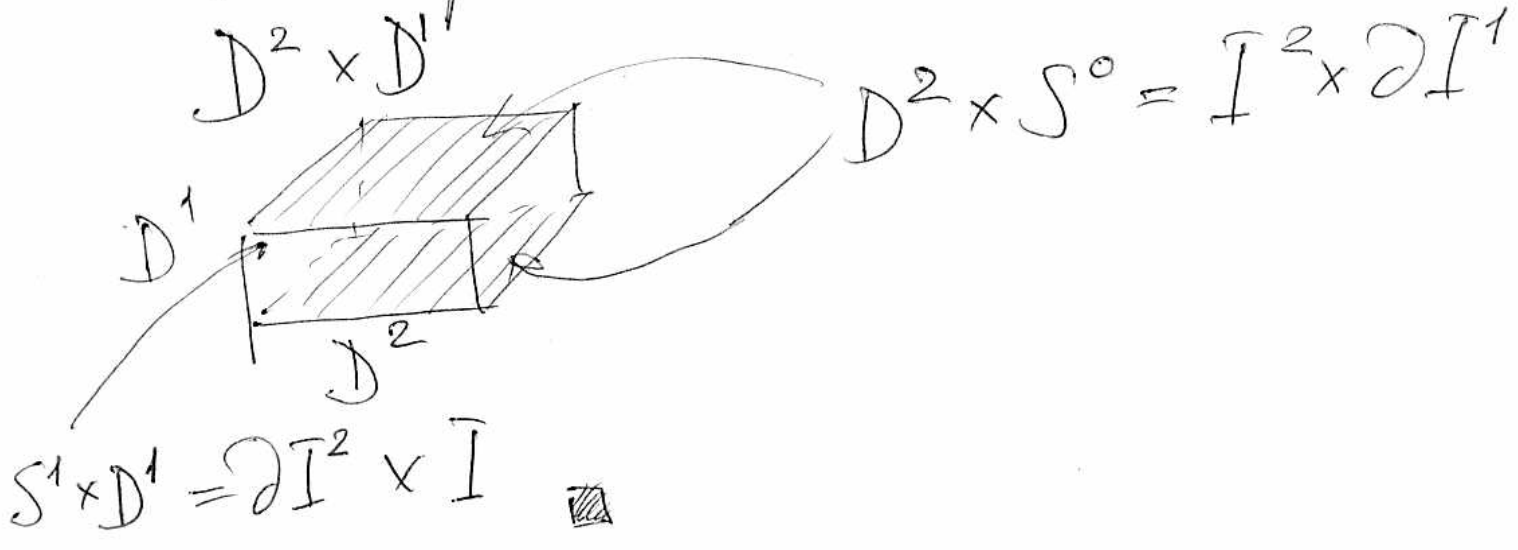
Lemma: $(D^p \times D^q, D^p \times S^{q-1} \cup S^{p-1} \times D^q) \cong (D^n, S^{n-1})$

Topological pairs. (2nd subspace of the 1st)

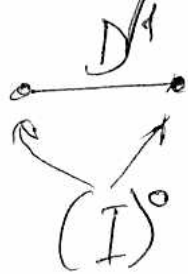
a pair $= (X, A)$ is just a top. space X and a subspace $A \subset X$.

There is a homeomorphism

Idea of proof: $D^n \cong I^n$



5. I CW complex



$X \times I$ CW complex containing two copies of X and one $(n+1)$ -cell for each n -cell of X .

Def. A cellular homotopy is a cellular map $h: X \times I \rightarrow Y$ $h|_{X \times [0]} \sim h|_{X \times [1]}$

\Rightarrow Homotopy category of CW complexes!

Objects: CW complexes
 Morphism: cellular homotopy classes of cellular maps.

CW pair: (X, A) X CW complex
 A subcomplex

morphisms of CW pairs $(X, A) \xrightarrow{f} (Y, B)$
 $f: X \rightarrow Y$ cellular
 $f(A) \subset B$

cellular homotopy for pairs!

$(X, A) \times I \xrightarrow{\sim} (Y, B)$

$(X \times I, A \times I) \Rightarrow$ Homotopy category of CW pairs

3 Axiomatic Homology Theory

Fix an abelian group G and consider
 (X, A) CW pairs (Homotopy category
of CW pairs)

X CW complex $\rightsquigarrow (X, \emptyset)$ CW pair
(\emptyset is also CW complex)

Theorem: $\forall q \in \mathbb{Z} \exists$ functors $H_q(X, A; G)$
from the homotopy category of pairs of
CW complexes to the category of abelian
groups together with a natural trans-
formation: $\partial: H_q(X, A; G) \rightarrow H_{q-1}(A; G) =$
 $H_{q-1}(A, \emptyset; G)$

These functors and natural transformations
satisfy and are determined by axioms!

Dimension: $X = \{*\} \Rightarrow H_0(X; G) = G$

Exactness: sequence $\forall (X, A) \text{ CW pair } H_q(X; G) = 0 \forall q \neq 0$
(unbounded on the left) \mathbb{Z}

$\dots \rightarrow H_q(A; G) \rightarrow H_q(X; G) \rightarrow H_q(X, A; G)$

$\xrightarrow{\partial} H_{q-1}(A; G) \rightarrow \dots$ is exact.

natural maps and connecting homeomorphisms
+ Excision & Additivity / read Category Theory