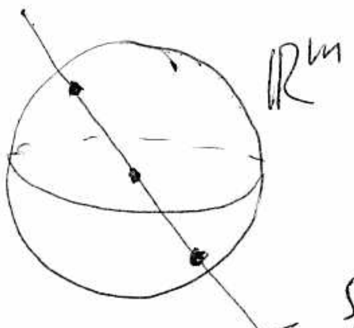
Math 8306 Lecture 3 (09.13.2004)

CW Complexes

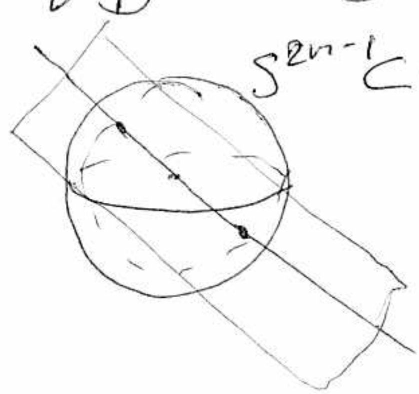
More examples:

1. $\mathbb{R}P^n = \{\text{lines in } \mathbb{R}^{n+1}\}$ contains one m-cell for each $m \le n$, attached via
$\partial D^m = S^{m-1} \to \mathbb{R}P^{m-1}$ (standard double cover)



2. $\mathbb{C}P^n = \{\text{complex lines in } \mathbb{C}^{n+1}\}$ contains one $2m$-cell for each m : ~~2~~$m \le n$ attached via
$\partial D^{2m} = S^{2m-1} \to \mathbb{C}P^{m-1}$

$S^{2m-1} \subset \mathbb{C}^m$ $\qquad$ $(2m-1)$-skeleton

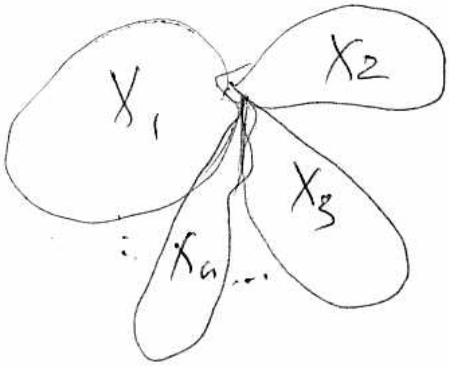


fiber bundle
fiber - circle
пересечение линии в $\mathbb{C}^m$
со сферой

"fiber bundle with fiber" S^1

Operations with CW complexes

1. $\vee_i X_i = \underbrace{\coprod_i}_{\text{disjoint union}} X_i \Big/ x_i^0 \sim x_j^0,\ \forall i,j$, where x_i^0 is a basepoint chosen at a vertex of X_i (which is a CW cx)



wedge = bouquet

has one m-cell for each cell of X_i and each i ($m \geq 1$)

2. X, Y CW complexes, $A \subset X$ subcomplex
$f: A \to Y$ cellular (preserves the skeleton)

Then $Y \cup_f X$ is a CW complex containing Y as a subcomplex and one cell for each cell of X not in A.
$(Y \cup_f X)/Y \cong X/A$ (isomorphism of CW complexes)

3. Colimit (here: union) (direct limit, inductive limit / inverse limit)

$X_n \subset X_{n+1}$ (inclusion as a subcomplex)
Then $\bigcup_n X_n$ is a CW complex, containing X_n as a subcomplex.

4. X, Y CW complexes

$X \times Y$ a CW complex with an n-cell for each pair (p-cell of X, q-cell of Y) for $p+q=n$

Lemma: $\left(D^p \times D^q,\ D^p \times S^{q-1} \cup S^{p-1} \times D^q\right) \cong \left(D^n, S^{n-1}\right)$

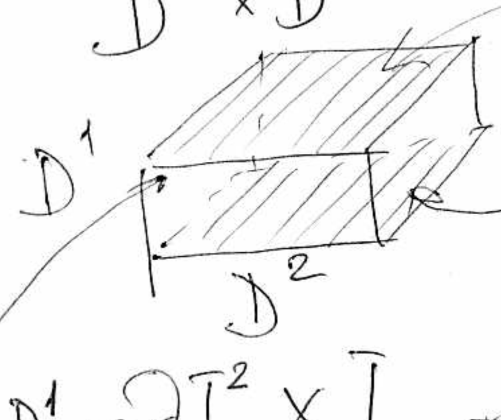
Topological pairs. (2nd subspace of the 1st)

a pair $= (X, A)$ is just a top. space X and a subspace $A \subset X$.

There is a homeomorphism ......

Idea of proof: $D^n \cong I^n$

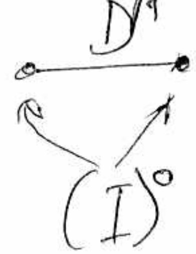
$D^2 \times D^1$



$D^2 \times S^0 = I^2 \times \partial I^1$

$S^1 \times D^1 = \partial I^2 \times I$ ∎

5. I CW complex



$X \times I$ CW complex containing two copies $X[0]$ and $X[1]$ of X and one $(n+1)$-cell for each n-cell of X.

<u>Deff.</u> A <u>cellular homotopy</u> is a cellular map $h: X \times I \to Y$ $\quad h|_{X[0]} \sim h|_{X[1]}$

$\Rightarrow$ <u>Homotopy category of CW complexes</u>:

Objects: CW complexes

Morphism: cellular homotopy classes of cellular maps.

CW pair: (X, A) $\quad X$ CW complex, A subcomplex

morphisms of CW pairs $(X, A) \xrightarrow{f} (Y, B)$

$f: X \to Y$ cellular

$f(A) \subset B$

cellular homotopy for pairs:

$$\underbrace{(X, A) \times I}_{(X \times I, A \times I)} \xrightarrow{f} (Y, B)$$

$\Rightarrow$ <u>Homotopy category of CW pairs</u>

Axiomatic Homology Theory

Fix an abelian group G and consider (X, A) CW pairs (Homotopy category of CW pairs).

X CW complex $\rightsquigarrow (X, \emptyset)$ CW pair

($\emptyset$ is also CW complex)

<u>Theorem</u>: $\forall q \in \mathbb{Z}$ $\exists$ functors $H_q(X, A; G)$ from the Homotopy cathegory of pairs of CW complexes to the category of abelian groups together with a natural transformation: $\partial : H_q(X, A; G) \to H_{q-1}(A; G) =$

$$= H_{q-1}(A, \emptyset; G)$$

These functors and natural transformations satisfy and are determined by axioms:

<u>Dimension</u>: $X = \{*\} \Rightarrow H_0(X; G) = G$, $H_q(X; G) = 0$ $\forall q \neq 0 \in \mathbb{Z}$

<u>Exactness</u>: sequence $\forall (X,A)$ CW pair (unbounded on the left)

$$\ldots \to H_q(A; G) \to H_q(X; G) \to H_q(X, A; G) \xrightarrow{\partial} H_{q-1}(A; G) \to \ldots$$ is exact.

natural maps and connecting homomorphism

+ Excision & Additivity / read Category theory