Math 8306 Lecture 13 (W 10.06.2004)

Singular Homology is a Homology Theory

Homotopy Axiom: $f \sim g : \begin{matrix} X \to Y \\ \cup \quad \cup \\ A \to B \end{matrix}$

$$\Rightarrow f_* = g_* : H_\cdot(X, A; G) \longrightarrow H_\cdot(Y, B; G)$$

Step 1: enough to show for $f, g : X \to Y, f \sim g$

$i_* = j_*$ for $i, j : X \to X \times I$

$i(x) = (x, 0)$

$j(x) = (x, 1)$

Step 2 $H_n(C; G) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$ $\Big|$ $(S_c)_n : C_n(C; G) \to C_{n+1}(C, G)$, $c \in C$

if C is convex in $\mathbb{R}^N$

Step 3 Inductive construction of ~~s_n~~

$s_n : C_n(X) \to C_{n+1}(X \times I)$

satisfying

(i) $ds_n + s_{n-1}d = j_* - i_*$ (i.e., $s_\cdot$ is a chain homotopy)

(ii) (naturality): $\forall f : X \to Y$

$$\begin{tikzcd} C_n(X) \arrow[r, "s_n"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & C_{n+1}(X \times I) \arrow[d, "(f \times id)_*"] \\ C_n(Y) \arrow[r, "s_n"] & C_n(Y \times I) \end{tikzcd}$$

$$id_{\Delta^n} : \Delta^n \to \Delta^n \in C_n(\Delta^n)$$
$$T : \Delta^n \to X \in C_n(X)$$
$$T = T_*(id_{\Delta^n}) = T \circ id_{\Delta^n} = T$$

$$\begin{tikzcd} & \Delta \arrow[r, "T"] & X \\ \Delta \arrow[ur, "id_{\Delta^n}"] & & \end{tikzcd}$$

$$\begin{tikzcd} id_{\Delta^n} \in \arrow[d] & C_n(\Delta^n) \arrow[r, "s_n"] \arrow[d, "T_*"] & C_{n+1}(\Delta^n \times I) \arrow[d, "(T\times id)_*"] \\ T \in & C_n(X) \arrow[r, "s_n"] \arrow[d, "f_*"] & C_{n+1}(X \times I) \\ & C_n(Y) & \end{tikzcd}$$

$f_* T_*$

$\to s_n(T) = (s_n T_*) id_{\Delta^n} = (T \times id)_* s_n$

Thus, if $s_n(id_{\Delta^n})$ is defined, the naturality forces us to define $s_n(T)$ uniquely from this diagram.

Thus, need to define $S_n(id_{\Delta^n})$

$n < 0$ $\quad s_n(id_{\Delta^n}) = 0 \quad : C_n(\Delta^n) = 0 \to C_{n+1}(\Delta^n \times I)$

$n = 0$ $\quad S_0(id_{\Delta^0}) = id_I \in C_1(\Delta^0 \times I)$ $\quad (I = \Delta^1)$.

$$ds_0 + s_{-1}d \overset{?}{=} j_* - i_*$$

$$ds_0(id_{\Delta^0}) = j_*(id_{\Delta^0}) - i_*(id_{\Delta^0})$$
$$\| $$
$$d(id_I) = \left\{\begin{matrix}\Delta^0 \to I \\ * \to 1\end{matrix}\right\} - \left\{\begin{matrix}\Delta^0 \to I \\ * \to 0\end{matrix}\right\}$$

$$\Delta^1 \to I = \Delta^1$$

Assume we have constructed $s_k(id_{\Delta^k})$ for $k<n$

Construct $s_n(id_{\Delta^n})$

$$ds_{n-1} + s_{n-2}d = j_* - i_*$$

$$ds_{n-1}d + \underbrace{s_{n-2}d^2}_{=0} = j_*d - i_*d = dj_* - di_*$$

(i, j commute with d).

$$d\big(\underbrace{s_{n-1}d(id_{\Delta^n}) - j_*(id_{\Delta^n}) + i_*(id_{\Delta^n})}\big) = 0$$

$\Rightarrow \in$ Kernel (d).

$s_{n-1}d(id_{\Delta^n}) - j_*(id_{\Delta^n}) + i_*(id_{\Delta^n}) \in \ker d$, $d: C_n(\Delta^n \times I) \to C_{n-1}(\Delta^n \times I)$

By step 2, $\exists z \in C_{n+1}(\Delta^n \times I)$ such that $\quad T: \Delta^n \to \Delta^n \times I \in C_n(\Delta^n \times I)$

$$dz = s_{n-1}d(id_{\Delta^n}) - j_*(id_{\Delta^n}) + i_*(id_{\Delta^n})$$

$$s_n(id_{\Delta^n}) := -z$$

$$ds_n(id_{\Delta^n}) + s_{n-1}d(id_{\Delta^n}) = j_*(id_{\Delta^n}) - i_*(id_{\Delta^n}).$$

By naturality, $ds_n + s_{n-1}d = j_\cdot - i_\cdot$ will be satisfied for all $T \in C_n(X)$.

<u>Step 4</u> For a pair (X, A) use (ii) to show that $(s_A)_n$ and $(s_X)_n$

$$\begin{array}{ccc} A & C_n(A;G) \xrightarrow{(s_A)} & C_{n+1}(A \times I, G) \\ \downarrow & \downarrow & \downarrow \\ X & C_n(X;G) \xrightarrow{(s_X)_n} & C_{n+1}(X \times I; G) \end{array}$$

induce a chain homotopy

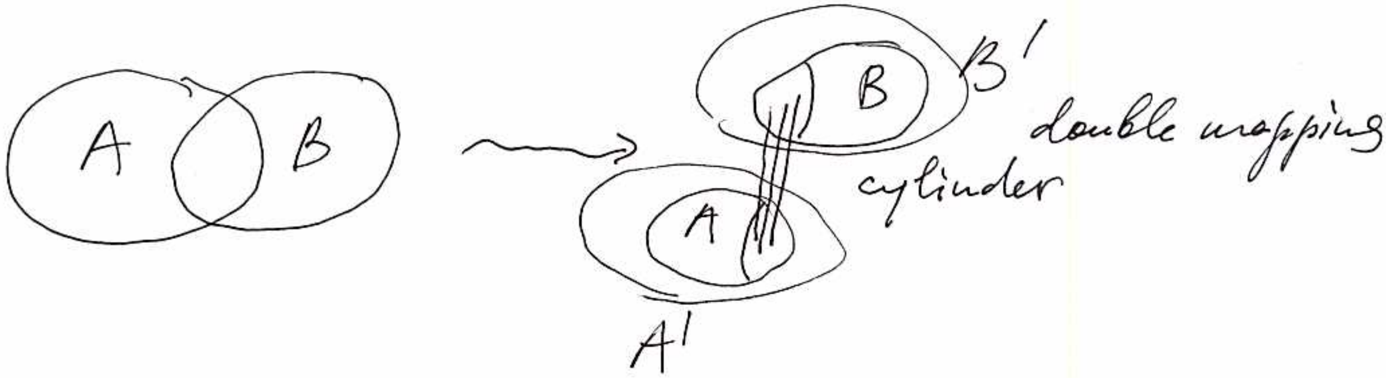
$C_n(X,A;G) \xrightarrow{(s_{X,A})_n} C_{n+1}(X\times I, A\times I; G)$. ∎

Excision

$X = A\cup B$ A, B open in X

$H_\cdot(A, A\cap B) \xrightarrow{\simeq} H_\cdot(X,B)$

CW complexes: $X = A\cup B$, A, B subcomplexes of a CW complex X (in particular, closed)



Tor G = torsion in G

$\overset{?}{=} \mathrm{Tor}_{\mathbb{Z}}(G, \mathbb{Q})$