Math 8306 Lecture 10 (W. 09.29.04)

Additivity Axiom for Singular Homology

$$X = \coprod_{\alpha \in A} X_\alpha$$

$$\bigoplus_{\alpha\in A} H_\bullet(X_\alpha; G) \xrightarrow{\sim} H_\bullet(X; G)$$

$$S_n X = \{T : \Delta^n \to X\} \underset{\uparrow}{=} \coprod_{\alpha \in A} \{T : \Delta^n \to X_\alpha\}$$

because $T(\Delta^n)$ is path connected
image of Δ^n

$$= \coprod_{\alpha\in A} S_n X_\alpha$$

$\big\downarrow$ $\cancel{\mathbb{Z}}$-free abelian groups

$$C_n(X) = \bigoplus_{\alpha\in A} C_n(X_\alpha)$$

$$C_n(X) \otimes G = \bigoplus_{\alpha\in A} \left(C_n(X_\alpha) \otimes G\right) \overset{H_\bullet}{\Longrightarrow} H_n(X;G) =$$

$$= \bigoplus_\alpha H_n(X_\alpha; G)$$

Homotopy Axiom for Singular Homology

$H : X \times I \to Y$ $\quad$ $H(x,0) = f(x)$, $\quad H(x,1) = g(x)$ $\quad$ $H : f \underset{X \to Y}{\simeq} g$

$H_\bullet(X) \to H_\bullet(Y)$ $\quad$ (for today $H_\bullet(X) = H_\bullet(X; G)$)

Then $f_* = g_*$

For pairs (X, A):

$$\begin{array}{ccc} H : X \times I & \longrightarrow & Y \\ \cup & & \cup \\ A \times I & \longrightarrow & B \end{array}$$

Then $f_* = g_* : H_\bullet(X,A) \to H_\bullet(Y,B)$

Reminder:

$I_\bullet : 0 \to \mathbb{Z}[I] \to \mathbb{Z}[0] \oplus \mathbb{Z}[1] \to 0.$

complex

$C_\bullet \otimes I_\bullet \xrightarrow{H} D_\bullet \iff$ homotopy $s : C_\bullet \to D_{\bullet+1}$

$$sd + ds = g - f$$

$H(x \otimes [0]) = f(x)$

$H(x \otimes [1]) = g(x)$

$f, g : C_\bullet \to D_\bullet$

$$\Downarrow$$

$$g_* = f_* : H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$$

$[x] \in \ker d_C / \operatorname{Im} d_C = H_\bullet(C_\bullet)$

$x \in \ker d_C$

$g_*(x) - f_*(x) = g(x) - f(x)$

$= sdx + dsx = dsx \in \operatorname{Im} d_D = 0 \in H_\bullet(D_\bullet)$

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_C"] \arrow[d, "f,g"] & C_n \arrow[r, "d_C"] \arrow[d, "f,g"] \arrow[dl, "s"'] & C_{n-1} \arrow[r] \arrow[d, "f,g"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d_D"] & D_n \arrow[r, "d_D"] & D_{n-1} \arrow[r] & {} \end{tikzcd}$$

Plan 1 of proof: 1. $C_\bullet(X\times I) = C_\bullet(X)\otimes C_\bullet(I)$

$= 0 \in H_\bullet(D_\bullet)$

To avoid topologizing

$$C_\bullet(X\times I) \underset{AW}{\overset{EZ}{\leftrightarrows}} C_\bullet(X)\otimes C_\bullet(I)$$

Nontrivial Prove that this is homotopy equivalence

2. $C_\bullet(I) \simeq I_\bullet$ Prove that this is a h. equivalence (straightforward)

3. $H: X\times I \to Y$ $H(x,0)=f(x), H(x,1)=g(x)$.

$$\begin{array}{c} H_*: C_\bullet(X\times I) \to C_\bullet(Y) \\ \wr\!\uparrow \quad\quad \nearrow_{H_{alg}} \\ C_\bullet(X)\otimes I_\bullet \end{array}$$

$H_{alg}: C_\bullet(X)\otimes I_\bullet \to C_\bullet(Y)$ $H_{alg}(x\otimes[0]) = f_*(x)$, $H_{alg}(x\otimes[1]) = g_*(x)$

$\Rightarrow f_* = g_*$, on homology $H_\bullet(X)\to H_\bullet(Y)$

4. For pairs:

$$\begin{array}{ccc} C_\bullet(X)\otimes I_\bullet & \to & C_\bullet(Y) \\ \cup & & \cup \\ C_\bullet(A)\otimes I_\bullet & \to & C_\bullet(B) \end{array}$$

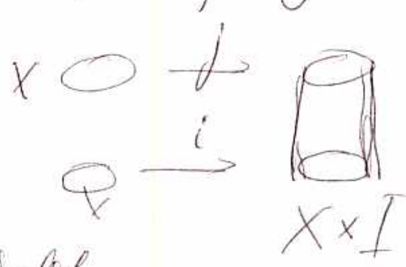
$(C_\bullet(X)/C_\bullet(A))\otimes I_\bullet \to C_\bullet(Y)/C_\bullet(B)$

$\Downarrow$

$f_* = g_*: H_\bullet(X,A) \to H_\bullet(Y,B)$

Plan 2 of the proof: $H: X\times I \to Y \Rightarrow f_* = g_*: H_\cdot(X)\to H_\cdot(Y)$

Step 1: Let $i: X \to X\times I$
$j: X \to X\times I$



Then $f = Hi$, $g = Hj$ $\xRightarrow{\text{functoriality}}$ $f_* = H_* i_*$, $g_* = H_* j_*$

Suffices to show that $i_* = j_*$

Step 2. Show $H_n(\Delta^k) = H_n(*) = \begin{cases} G & n=0\\ 0 & n\neq 0\end{cases}$

$H_n(C) = H_n(*)$ for any convex $C\subset\mathbb{R}^N$ (or starlike)

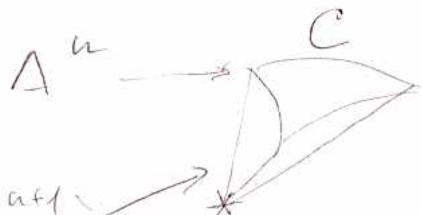
idea: construct alg. homotopy $s: C_n(C)\to C_{n+1}(C)$

$ds+sd = id-\varepsilon$ $\quad \varepsilon(T) = \begin{cases} T & \text{if } T: \Delta^0 = * \to * \\ 0 & \text{otherwise}\end{cases}$

$\varepsilon: C_\cdot(C)\to C_\cdot(C)$ (easy to show that ε is a chain map).

Exercise: $ds+sd = id-\varepsilon \Rightarrow H_\cdot(C) \cong H_\cdot(*)$

$s: C_n(C) \to C_{n+1}(C)$



$ds+sd \overset{?}{=} id-\varepsilon$ alt. Σ side bdry of sT $= T$

$dsT + sdT = T - \varepsilon(T)$

alternated Σ boundary of sT

$\Delta^0 \xrightarrow{T}$ sT $*$ $\quad dsT + 0 = T - \varepsilon$