

Additivity Axiom for Singular Homology

$$X = \coprod_{\alpha \in A} X_\alpha$$

$$\bigoplus_{\alpha \in A} H_n(X_\alpha; G) \cong H_n(X; G)$$

$$S_n X = \left\{ T: \Delta^n \rightarrow X \right\} = \coprod_{\alpha \in A} \left\{ T: \Delta^n \rightarrow X_\alpha \right\}$$

because $T(\Delta^n)$ is path connected image of Δ^n .

$$= \coprod_{\alpha \in A} S_n X_\alpha$$

$\left\{ \begin{array}{l} \text{abelian} \\ \text{free groups} \end{array} \right.$

$$C_n(X) = \bigoplus_{\alpha \in A} C_n(X_\alpha)$$

$$C_n(X) \otimes G = \bigoplus_{\alpha \in A} (C_n(X_\alpha) \otimes G) \xrightarrow{H_n} H_n(X; G) = \bigoplus_{\alpha \in A} H_n(X_\alpha; G)$$

Homotopy Axiom for Singular Homology

$$H: X \times I \rightarrow Y$$

$$H(x, 0) = f(x)$$

$$H(x, 1) = g(x)$$

$$H: f \circ g$$

$$H_*(X) \rightarrow H_*(Y)$$

for today $H_*(X) = H_*(X; G)$

Then $f_+ = g_+$

For pairs (X, A) :

$$M: \begin{array}{ccc} X \times I & \rightarrow & Y \\ \cup & & \cup \\ A \times I & \rightarrow & B \end{array}$$

Then $f_* = g_* : H_*(X, A) \rightarrow H_*(Y, B)$

Reminder:

$$I_* : 0 \rightarrow \mathbb{Z}[I] \rightarrow \mathbb{Z}[0] \oplus \mathbb{Z}[1] \rightarrow 0.$$

complex

$$C_* \otimes I_* \xrightarrow{M} D_* \Leftrightarrow \text{homotopy } s : C_* \rightarrow D_{*+1}$$

$$H(x \otimes [0]) = f(x)$$

$$H(x \otimes [1]) = g(x)$$

$$sd + ds = \overline{g} - f$$

$$f, g : C_* \rightarrow D_*$$

\Downarrow

$$g_* = f_* : H_*(C_*) \rightarrow H_*(D_*)$$

$$[x] \in \ker d_C / \text{Im } d_C = H_*(C_*)$$

$x \in \ker d_C$

$$g_*^*(x) - f_*^*(x) = g(x) - f(x)$$

$$= sd(x) + ds(x) = ds(x) \in \text{Im } d_D = 0 \in H_*(D_*)$$

$$\begin{array}{ccccccc} \dots & \rightarrow & C_{n+1} & \xrightarrow{d_C} & C_n & \xrightarrow{d_C} & C_{n-1} & \rightarrow \dots \\ & & \downarrow f, g & \swarrow s & \downarrow f, g & \swarrow f, g & \downarrow f, g & \\ \dots & \rightarrow & D_{n+1} & \xrightarrow{d_D} & D_n & \xrightarrow{d_D} & D_{n-1} & \rightarrow \dots \end{array}$$

Plan 1 of proof : 1. $C_*(X \times I) = C_*(X) \otimes C_*(I)$
 $= 0 \in H_*(\mathbb{D})$

To avoid topologizing

$$C_*(X \times I) \begin{matrix} \xleftarrow{EZ} \\ \xrightarrow{AW} \end{matrix} C_*(X) \otimes C_*(I)$$

Nontrivial \Rightarrow Prove that this is a homotopy equivalence

2. $C_*(I) \xrightarrow{\sim} I$. Prove that this is a h. equivalence (straightforward)

3. $H: X \times I \rightarrow Y$ $H(x,0) = f(x), H(x,1) = g(x)$.

$$H_*: C_*(X \times I) \rightarrow C_*(Y)$$

$$\begin{matrix} \uparrow \\ C_*(X) \otimes I \end{matrix} \xrightarrow{H_{alg}}$$

$$H_{alg}: C_*(X) \otimes I \rightarrow C_*(Y)$$

$$\begin{matrix} H_{alg}(x \otimes [0]) = f_*(x) \\ H_{alg}(x \otimes [1]) = g_*(x) \end{matrix}$$

$\Rightarrow f_* = g_*$, on homology $H_*(X) \rightarrow H_*(Y)$

4. For pairs : $C_*(X) \otimes I \rightarrow C_*(Y)$

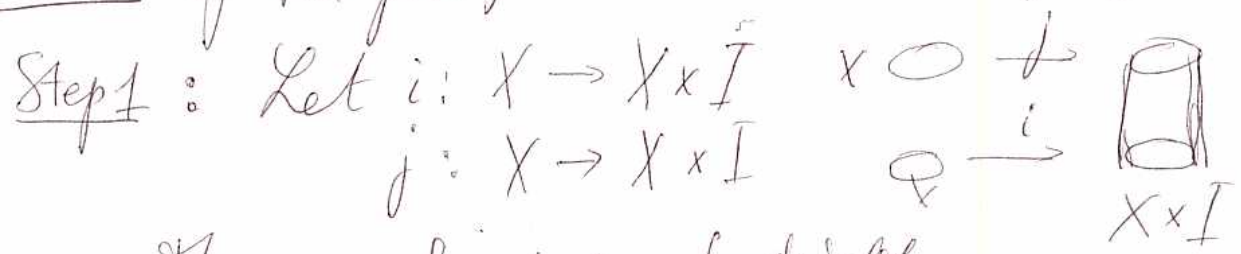
$$C_*(A) \otimes I \rightarrow C_*(B)$$

$$(C_*(X)/C_*(A)) \otimes I \rightarrow C_*(Y)/C_*(B)$$

$$\downarrow$$

$$f_* = g_*: H_*(X, A) \rightarrow H_*(Y, B)$$

Plan 2 of the proof: $M: X \times I \rightarrow Y \Rightarrow f_* = g_*: H_*(X) \rightarrow H_*(Y)$



Then $f = M \circ i$ *functoriality* $\Rightarrow f_* = M_* \circ i_*$
 $g = M \circ j$ $g_* = M_* \circ j_*$

Suffices to show that $i_* = j_*$

Step 2. Show $H_n(A^k) = H_n(*)$ for $n=0$
 0 $n \neq 0$

$H_n(C) = H_n(*)$ for any convex $C \subset \mathbb{R}^N$
 (or starlike)



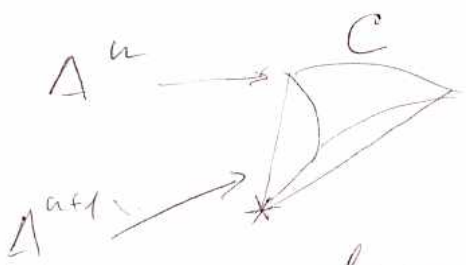
idea: construct alg. homotopy $s: C_n(C) \rightarrow C_{n+1}(C)$

$ds + sd = id - \epsilon$ $\epsilon(T) = \begin{cases} T & \text{if } T: \Delta^0 = * \rightarrow * \\ 0 & \text{otherwise} \end{cases}$

$\epsilon: C_0(C) \rightarrow C_1(C)$ (easy to show that ϵ is a

Exercise: $ds + sd = id - \epsilon \Rightarrow H_*(C) \cong H_*(*)$ (chain map)

$s: C_n(C) \rightarrow C_{n+1}(C)$



$ds + sd = id - \epsilon$ \Rightarrow alt. Σ side body of $sT = T$

$\rightarrow dsT + sdT = T - \epsilon(T)$
 alternated Σ boundary of sT

