Math 8306 Lecture 1

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Homology (Ch 2 Hatcher)

AT (algebraic topology): $X \longrightarrow A(X)$

X: top. space $\in$ Top

$A(X)$: algebraic object — v. space, group, ring, etc. $\in$ Algebra

$A(X)$ homotopy invariant

$X \xrightarrow{f} Y \rightsquigarrow A(f): A(X) \to A(Y)$

A - functor from Top to Alg.

if f is homotopic to g $f \sim g \Rightarrow A(f) = A(g)$ (homotopy invariance)

$I = [0,1]$ $\quad X \times I \xrightarrow{h} Y \quad$ $h(x,t)$ is homotopy between f and g, $t \in I$

$h(-,0) = f$

$h(-,1) = g$

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Ultimate goal of AT:

come up with $A(X)$ such that it determines X (up to homotopy).

In practice

want $A(X)$ to be computable

$X \rightsquigarrow A(X)$, compute $A(X)$ for good enough spaces X

and then solves problems about X, using $A(X)$

[How to hear the shape of the drum? spectrum of the laplace operator

shape - X $\searrow A(X)$]

or $f: X \to Y \rightsquigarrow A(f)$ and do the same

Fundamental Theorem of Algebra

Theorem: $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$. Then $f(x)$ has at least one complex root.

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<u>Proof</u>: $f: \mathbb{C} \to \mathbb{C}$

$\cup$

$\tilde{f}: S^1 \to S^1$

$A(X)$ is $\pi_1(X)$ = fundamental group.

$\pi_1(S^1) = \mathbb{Z}$

abelian group of integers

Assume $f(x) \neq 0 \quad \forall x \in \mathbb{C}$

$$\tilde{f}(x): S^1 \longrightarrow S^1$$

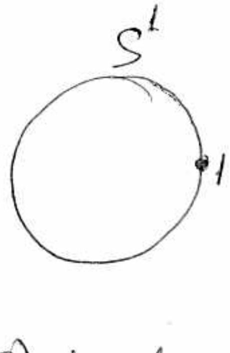
$$\| \quad \frac{f(x)}{|f(x)|}$$

continuous

Compute $\deg \tilde{f}$ in 2 ways

~~Definition of~~ <u>deg</u> of a map

$$\pi_1(S^1, 1) \xrightarrow{\tilde{f}_*} \pi_1(S^1, \tilde{f}(1))$$



basepoint

$$\xrightarrow{\gamma[a]} \pi_1(S^1, 1)$$

where a is path from $f(1)$ to 1

$$\gamma[a](b) = aba^{-1}$$

$b \in \pi_1(S^1, \hat{f}(1))$

$\gamma^{[a]}$ is independent of the choice of a.

$i \in \pi_1(S^1, 1)$ a generator maps to $\deg \hat{f} \cdot i$ by the map $\gamma^{[a]} \circ f_* : \mathbb{Z} \to \mathbb{Z}$

1st way $f(x) \neq 0$ for $|x| \leq 1$

Define a homotopy $h(x,t) := \frac{f(xt)}{|f(xt)|}$

$h : S^1 \times I \to S^1$

$h(0) = \frac{f(0)}{|f(0)|}$ constant map $S^1 \to S^1 \Rightarrow$
$\Rightarrow \deg = 0$.

$h(1) = \hat{f} \qquad \Rightarrow \deg \hat{f} = 0$.

2nd way: $f(x) \neq 0$ for $|x| \geq 1$

Define $H(x,t) = \frac{f(x/t)}{|f(x/t)|} = \frac{t^n f(x/t)}{|t^n f(x/t)|}$

$H : S^1 \times I \to S^1$

$$t^n f(x/t) = t^n\left(\left(\frac{x}{t}\right)^n + a_{n-1}\left(\frac{x}{t}\right)^{n-1} + \dots + a_0\right) =$$

$$= x^n + a_{n-1} x^{n-1} t + \dots a_0 t^n.$$

$$H(0) = \frac{x^n}{|x|^n} = x^n$$

$$H(1) = \tilde{f}$$

$$\deg(x \mapsto x^n) = n \Rightarrow \deg \tilde{f} = n$$

Thus $n = 0$.

1. Sato ~~introduction~~ Alg. to topology
   An intuitive approach.

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