

Replace in HW2 the problem on  $H_*(\mathbb{R}P^n; \mathbb{Z}_2)$  with  $H_*(K; \mathbb{Z}_2)$

$$H_n(K; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z} & n=1 \\ 0 & n=2 \end{cases}$$

Künneth Formula

→ principal ideal domain

Theorem If  $R$  PID,  $C$ . flat complex of  $R$ -modules and  $D$ . any complex of  $R$ -modules. Then  $\forall n \in \mathbb{Z}$   $\exists$  natural short exact sequence (SES)

$$(1) \quad 0 \rightarrow \bigoplus_{p+q=n} H_p(C_i) \otimes H_q(D_i) \xrightarrow{\alpha} H_n(C_i \otimes D_i) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(C_i), H_q(D_i)) \rightarrow 0$$

$R\text{-mod} \subset \underbrace{\text{graded } R\text{-mod}}_{\substack{\text{category of} \\ R\text{-modules}}} \subset \text{complexes of } R\text{-mod-s}$   
 $D_i: \dots \rightarrow D_n \xrightarrow{d_n} D_{n-1} \rightarrow \dots$   
 $H_n(D_i) = D_n$

$$D \mapsto \bigoplus_n D_n \hookrightarrow \dots \xrightarrow{0} D_n \xrightarrow{0} D_{n-1} \xrightarrow{0} \dots$$

$$D_n = \begin{cases} 0 & n \neq 0 \\ D & n = 0 \end{cases}$$

$$0 \rightarrow D \rightarrow 0$$

1      0      -1

Sequence (1) splits but ~~not~~ naturally

Topology Theorem If  $X, Y$  are CW complexes,  
then  $C.cw(X \times Y) \cong C.cw(X) \otimes C.cw(Y)$

$\Rightarrow$  Künneth Theorem:  $H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \bigoplus_{p+q=n-1} \text{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y))$

$\mathbb{Z}$ -principal ideal domain

For arbitrary spaces  $X, Y$ :

$$C.(X \times Y) \xrightarrow{\text{h. equiv.}} C.(X) \otimes C.(Y)$$

Remark

$C., D.$  are complexes of ab. groups,  $R$  a commutative ring

$$C. \otimes_{\mathbb{Z}} D. \otimes_{\mathbb{Z}} R \cong C. \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} (D. \otimes_{\mathbb{Z}} R)$$

as  $R$ -modules

$$R \otimes_R (\dots) = (\dots)$$

Corollary If  $R$  is a field, then we have a natural isomorphism

$$\bigoplus_{p+q=n} H_p(C.) \otimes H_q(D.) \xrightarrow{\cong} H_n(C. \otimes D.)$$

or  $H. (C.) \otimes H. (D.) \xrightarrow{\cong} H. (C. \otimes D.)$

Proof of the corollary (with indication where Tor's will show up in theorem)

Step 1: Assume  $C. = C_p$ ,  $d = 0$ ,  $\begin{pmatrix} 0 & -D & 0 \\ & 0 & - \end{pmatrix}$   
 $C_p$  is a flat module

$$\begin{array}{ccccccc}
 & & & & & & 28 \\
 0 & \rightarrow & B_q(D) & \rightarrow & Z_q(D) & \rightarrow & H_q(D) \rightarrow 0 \\
 & & \uparrow d_{q+1} & \curvearrowright & \downarrow & & \\
 & & D_{q+1} & \rightarrow & D_q & & \\
 & & & & \downarrow d_q & & \\
 & & & & D_{q-1} & & 
 \end{array}$$

$$\begin{aligned}
 Z_q(D) &= \text{Ker } d_q \\
 D_{q+1} &\xrightarrow{d_{q+1}} D_q \xrightarrow{d_q} D_{q-1} \\
 B_q(D) &= \text{Im } d_{q+1} \\
 &\subseteq Z_q(D)
 \end{aligned}$$

exact row & columns

Tensor this diagram with  $C_p$ ; same exactness properties  
 (so far no Tors show up even in the general case.)

For  $n = p + q$ , have  $Z_n(C_p \otimes D) = C_p \otimes Z_q(D)$

$$B_n(C_p \otimes D) = C_p \otimes B_q(D)$$

$\Downarrow$

$$H_n(C_p \otimes D) = C_p \otimes H_q(D)$$

Remark:

$$\text{Tor}_1^R(H_p(C), H_q(C)) = 0 \text{ because } H_p(C) = C_p \text{ is flat}$$

Step 2 General case:  $C \neq C_p$  for any  $p$ .