Math 8306 Lecture 28 (W 11.10.2004)

Replace in HW2 the problem on $H_\cdot(\mathbb{RP}^4; \mathbb{Z}_2)$ with $H_\cdot(K; \mathbb{Z}_2)$

$$H_n(K; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z} & n=1 \\ 0 & n=2 \end{cases}$$

<u>Künneth Formula</u>

<u>Theorem</u> If R PID (principal ideal domain), $C_\cdot$ flat complex of R-modules and $D_\cdot$ any complex of R-modules. Then $\forall n \in \mathbb{Z}$ $\exists$ natural short exact sequence (SES)

$$(1)\quad 0 \to \bigoplus_{p+q=n} H_p(C_\cdot) \otimes H_q(D_\cdot) \xrightarrow{\alpha} H_n(C_\cdot \otimes D_\cdot) \to$$
$$\to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_\cdot), H_q(D_\cdot)) \to 0$$

$\underset{\substack{\text{cathegory of} \\ R\text{-modules}}}{\underline{R\text{-mod}}} \subset \underbrace{\text{graded } R\text{-mod}}_{\bigoplus_{n\in\mathbb{Z}} D_n} \subset$ complexes of R-mod-s

$D \in R\text{-mod}$

$$D_\cdot : \cdots \to D_n \xrightarrow{d_n} D_{n-1} \to$$
$$H_n(D_\cdot) = D_n$$

$$D \longmapsto \bigoplus_n D_n \longmapsto \cdots \xrightarrow{0} D_n \xrightarrow{0} D_{n-1} \xrightarrow{0} \cdots$$

$$D_n = \begin{cases} 0 & n \neq 0 \\ D & n=0 \end{cases}$$

$$\underset{1}{0} \to \underset{0}{D} \to \underset{-1}{0}$$

Sequence (1) splits but not naturally

<u>Topology</u> <u>Theorem</u> If X, Y are CW complexes, then $C_\cdot^{CW}(X \times Y) \cong C_\cdot^{CW}(X) \otimes C_\cdot^{CW}(Y)$

$\Rightarrow$ Künneth Theorem: $H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus$ ($\mathbb{Z}$ - principal ideal doma

$$\oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y))$$

For arbitrary spaces X, Y:

$C_\cdot(X \times Y) \overset{\text{h. equiv.}}{\rightleftarrows} C_\cdot(X) \otimes_{\mathbb{Z}} C_\cdot(Y)$

<u>Remark</u>

$C_\cdot, D_\cdot$ are complexes of ab. groups, R a commutative ring

$C_\cdot \otimes_{\mathbb{Z}} D_\cdot \otimes_{\mathbb{Z}} R \cong C_\cdot \otimes_{\mathbb{Z}} R) \otimes_R (D_\cdot \otimes_{\mathbb{Z}} R)$ as R-modules

$$R \otimes_R (\ldots) = (\ldots)$$

<u>Corollary</u> If R is a field, then we have a natural isomorphism

$\bigoplus_{p+q=n} H_p(C_\cdot) \otimes H_q(D_\cdot) \xrightarrow{\alpha} H_n(C_\cdot \otimes D_\cdot)$

or $H_\cdot(C_\cdot) \otimes H_\cdot(D_\cdot) \xrightarrow{\alpha} H_\cdot(C_\cdot \otimes D_\cdot)$

<u>Proof of the corollary</u> (with indication where Tor's will show up in Theorem)

<u>Step 1</u>: Assume $C_\cdot = C_p$, $d = 0$, $\left(\underset{1}{0} - \underset{0}{D} - \underset{-1}{0} \right)$

C_p is a flat module

$$\begin{tikzcd}
0 \arrow[r] & B_q(D) \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & Z_q(D) \arrow[r] \arrow[d] & H_q(D) \arrow[r] & 0. \\
 & D_{q+1} \arrow[r] \arrow[u, "d_{q+1}"] & D_q \arrow[d, "d_q"] & & \\
 & & D_{q-1} & &
\end{tikzcd}$$

$Z_q(D) = \mathrm{Ker}\, d_q$

$D_{q+1} \xrightarrow{d_{q+1}} D_q \xrightarrow{d_q} D_{q-1}$

$B_q(D) = \mathrm{Im}\, d_{q+1} \subset Z_q(D)$

exact row & columns

Tensor this diagram with C_p; same exactness properties (so far no Tor's show up even in the general case.)

For $n = p+q$, have $Z_n(C_p \otimes D.) = C_p \otimes Z_q(D.)$

$$B_n(C_p \otimes D.) = C_p \otimes B_q(D.)$$

$$\Downarrow$$

$$H_n(C_p \otimes D.) = C_p \otimes H_q(D.)$$

Remark: $\mathrm{Tor}_1^R(H_p(C.), H_q(C.)) = 0$ because $H_p(C.) = C_p$ is flat

Step 2 General case: $C. \neq C_p$ for any p.