Math 8306 Lecture 26 (F 11.05.2004)

Homology of $\mathbb{RP}^n$ with $\mathbb{Z}_2$ coefficients

$$C_\cdot^{cw}(\mathbb{RP}^n):\ 0\to\mathbb{Z}\to\dots\xrightarrow{0}\mathbb{Z}\xrightarrow{2}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to\dots\xrightarrow{2}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to 0$$

$$C_\cdot^{cw}(\mathbb{RP}^n;\mathbb{Z}_2) := C_\cdot^{cw}(\mathbb{RP}^n)\otimes\mathbb{Z}_2:$$

$$0\to\mathbb{Z}_2\xrightarrow{0}\mathbb{Z}_2\xrightarrow{0}\dots\to\mathbb{Z}_2\xrightarrow{0}\mathbb{Z}_2\to 0 \qquad 2\equiv 0 \bmod 2$$

$$R\otimes_R M = M$$

$$r\otimes m \longmapsto r\cdot m$$

$$1\otimes m \longleftarrow m$$

$$H_\cdot(\mathbb{RP}^n,\mathbb{Z}_2):\ \underset{n}{\mathbb{Z}_2}\ \underset{n-1}{\mathbb{Z}_2}\ \mathbb{Z}_2\cdots\underset{1}{\mathbb{Z}_2}\ \underset{0}{\mathbb{Z}_2}$$

$$H_\cdot(X) = H_\cdot(X;\mathbb{Z}) \xrightarrow{?} H_\cdot(X;G)$$

universal coefficients formula (Later)

Application of Homology

<u>Theorem</u>: There is no retraction r of n-dimensional disc D^n to its boundary

$$r: D^n \to S^{n-1}$$

$(n\geq 1)$, i.e., $S^{n-1}\hookrightarrow D^n\xrightarrow{r}S^{n-1}$ is $id_{S^{n-1}}$

<u>Proof</u>: Let's apply reduced homology

$$\underbrace{\tilde H_{n-1}(S^{n-1}) \to \underset{\overset{\parallel}{0}}{\tilde H_{n-1}(D^n)} \xrightarrow{r_*} \tilde H_{n-1}(S^{n-1})}_{id}$$

$\mathbb{Z} \to \mathbb{Z}$ is 0 and id at the same time $\blacksquare$

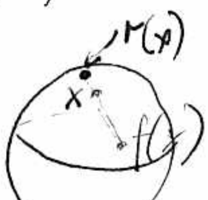
Remark: There is even no homotopy retraction
$r : D^n \to S^{n-1}$

Corollary (Brower Fixed Pt Theorem)

Any conts map $f : D^n \to D^n$ $(n \geq 0)$ has a fixed point

Proof: $n = 0$ trivial
$n \geq 1$
Suppose f doesn't have any fixed points
$x : f(x) = x$

D^n



Extend interval $f(x)$ to x through x and let $r(x)$ be the intersection with $S^{n-1} = \partial D^n$

$r : D^n \to S^{n-1}$ and $r(x) = x$ for $x \in S^{n-1}$
$\Rightarrow$ Thus contradiction

Back to universal coeff. formula

More Homological Algebra

Chains are rarely used to compute homology (beyond basic examples). One use them to proove algebraic theorems, that one uses for topological computations.

Goal: calculate $H_\cdot(X; G)$ from $H_\cdot(X)$
and $H_\cdot(X \times Y)$ from $H_\cdot(X)$ and $H_\cdot(Y)$

Assume R commuting ring (often $\mathbb{Z}$); $\otimes$ here is $\otimes_R$

Study complexes of R-modules:

$C_\cdot = (C_\cdot, d)$: $\cdots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \to \cdots$ $\quad d^2 = 0$

$C_\cdot, D_\cdot \quad H_\cdot(C_\cdot \otimes D_\cdot) = ?$ $\left[(C_\cdot \otimes D_\cdot)_n = \bigoplus_{p+q=n} C_p \otimes D_q \right.$

$d = d \otimes id + id \otimes d$

$\alpha: H_\cdot(C_\cdot) \otimes H_\cdot(D_\cdot) \longrightarrow H_\cdot(C_\cdot \otimes D_\cdot)$

$\alpha([a] \otimes [b]) := [a \otimes b]$, where a, b are cycles $\in \operatorname{Ker} d$ in $C_\cdot$ and $D_\cdot$ representing homology classes $[a], [b]$.

$d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$

where $a \in C_n$ and $|a| = n$

$a = da'$ $\quad d(a \otimes b) = (-1)^{|a|} da' \otimes db =$

$= -(1)^{|a|} d(a' \otimes db)$

First, treat special case $D_\cdot = M$ ($D_\cdot = 0 \to \underset{0}{M} \to 0$)

$\alpha: H_\cdot(C_\cdot) \otimes M \to H_\cdot(C_\cdot \otimes M)$.

<u>Def.</u> M is <u>flat</u> if functor $N \otimes M$ is exact (i.e. preserves the exact sequences in N).

N is a module

$0 \to N_1 \to N_2 \to N_3 \to 0$ exact

$0 \to N_1 \otimes M \to N_2 \otimes M \to N_3 \otimes M \to 0$.

Graded module $M_\cdot = \bigoplus_{n\in\mathbb{Z}} M_n$ is <u>flat</u>, if all M_n's are flat

<u>Fact</u> : In general, for a short exact sequence

$0 \to N_1 \to N_2 \to N_3 \to 0$

produces a long exact sequence

$\ldots \to \mathrm{Tor}_1^R(N_2, M) \to \mathrm{Tor}_1^R(N_3, M) \to N_1\otimes M \to N_2 \otimes M \to$
$\to N_3 \otimes M \to 0,$

where $\mathrm{Tor}_q^R(N, M)$ may be ~~defind~~ defined as follows (here: define only for R a PID):
construct a short exact sequence $0 \to F_1 \to F_0 \to M \to 0$ with F_0, F_1 free R-modules by taking any epimorphism $F_0 \xrightarrow{\varphi} M \to 0$ for a suitable free F_0.
The $F_1 := \operatorname{Ker}\varphi$ is ~~fre~~ free as a submodule of a free one over a PID.

$0 \to F_1 \to F_0 \to M \to 0$

and define $\mathrm{Tor}_1^R(N, M)$ to make the following exact:
$0 \to \mathrm{Tor}_1^R(N, M) \to N\otimes F_1 \to N \otimes F_0 \to N\otimes M \to 0.$

$\mathrm{Tor}_q^R(N, M) = 0$ for $q \geq 2$