Math 8306 Lecture 24 (M 11.01.2004)

Local Degree

$f: S^n \to S^n$, $n > 0$

Suppose for some point $y \in S^1$, its preimage $f^{-1}(y)$ is finite $= \{x_1, \dots, x_m\}$

Choose neibourhoods $U_1, \dots, U_m$ of these points and neighbd V of y, s.t. $f: U_i \to V$

$$f(U_i \setminus X_i) \subset V \setminus y$$

$$\left.\begin{array}{l} U_1 \ \bigcirc \\ U_2 \ \bigcirc \\ \vdots \\ U_m \ \bigcirc \end{array}\right\} \subset S^n$$

$$\downarrow f$$

$$V \ \bigcirc \subset S^n$$

Choose any i:

$H_n(S^n, S^n \setminus X_i)$ $\qquad H_\cdot(X, B) \xleftarrow{\simeq} H_\cdot(A, A \cap B)$

$X = A \cup B$ open

$X_i \quad S^n \setminus X_i$

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus X_i) \arrow[dl, "\text{excision}"'] \arrow[d, "k_i"] \arrow[r, "f_*"] & H_n(V, V \setminus y) \arrow[d, "\text{excision}"] \\
H_n(S^n, S^n \setminus X_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[ul, "\text{LES of pair, } H_q(S^n \setminus X_i) = 0 \text{ for } q>0"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\text{LES of pair}"']
\end{tikzcd}$$

$\mathbb{Z} \cong$

Outer vertical arrows: $H_n(U_i, U_i \setminus X_i) = H_n(S^n) = H_n(V, V \setminus y)$

Then the upper f_* is multiplied by an integer, called <u>local degree</u> at x_i: $\deg f|_{x_i}$

Claim: $\deg f = \sum_{i=1}^{m} \deg f|_{X_i}$

Remarks: 1. If f is a homeomorphism, then all maps in diagram are iso and $\deg f = \pm 1$ and $\deg f|_{X_i} = \pm 1$

2. If $f: U_i \to V$ is a homeomorphism, then again $\deg f|_{X_i} = \pm 1$

Proof: $H_n(S^n, S^n \setminus f^{-1}(y)) \cong \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus X_i)$

$A = \coprod U_i$, $B = S^n \setminus f^{-1}(y) = S^n \setminus (X_1, \dots, X_m)$

$H_n(\coprod U_i, \coprod U_i \setminus X_i)$

Comm. of upper left $\Delta \Rightarrow p_i$ followed by excis.

Choose $H_n(S^n) \cong \mathbb{Z}$

lower $\Delta \Rightarrow p_i j(1) = 1 \Rightarrow j(1) = (1, 1, \dots, 1) =$

$= \sum_{i=1}^{m} k_i(1)$

upper $\square \Rightarrow$ $f_* h_i(1) = \deg f_m|_{X_i}$

$\sum f_* k_i(1) = \sum_{i=1} \deg f|_{X_i}$

$\| \\ f_* j(1)$

lower $\square \Rightarrow f_* j(1) = \deg f$

$\square$