

Cellular Homology

$H_*(X) = H_*(X; \mathbb{Z})$  a homology theory for CW complexes  $X$

$$H_n^{CW}(X) = H_n(C_*^{CW}(X), d^{CW})$$

$$C_n^{CW}(X) = H_n(X^n, X^{n-1}) = \bigoplus_{\substack{\text{n-cells} \\ e_\alpha^n \text{ of } X}} \mathbb{Z} e_\alpha^n$$

$$d^{CW}(e_\alpha^n) = [\partial] - [-\partial]; \quad d^{CW}(e_\alpha^n) = \sum_{\beta} d_{\alpha\beta} e_\beta^{n-1}$$

$d_{\alpha\beta}: S^{n-1} = \partial D_\alpha^n \xrightarrow{\cong} X^{n-1} \rightarrow X^{n-1}/X^{n-2} = \bigvee_{\beta} S_\beta^{n-1}$

the degree of map  $\rightarrow S_\beta^{n-1}$

Def.  $k > 0, f: S^k \rightarrow S^k, H_k(S^k) \xrightarrow{f_*} H_k(S^k)$   
 $\mathbb{Z} \ni 1 \mapsto \text{deg } f \in \mathbb{Z}$

Theorem  $H_*^{CW}(X) = H_*(X; \mathbb{Z})$  (for the homology theory we started up with)

Corollary:  $H_*(X, A)^{CW} = H_*(X, A; \mathbb{Z})$

Five-Lemma with LES's

$$\begin{array}{ccccccccc} H_n^{CW}(A) & \rightarrow & H_n^{CW}(X) & \rightarrow & H_n^{CW}(X, A) & \rightarrow & H_{n-1}^{CW}(A) & \rightarrow & H_{n-1}^{CW}(X) \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ H_n(A) & \rightarrow & H_n(X) & \rightarrow & H_n(X, A) & \rightarrow & H_{n-1}(A) & \rightarrow & H_{n-1}(X) \end{array}$$

as soon as we have natural maps between  $H_*^{CW}$  and  $H_*$

Proof: 1. Claim  $H_q(X^n) = \begin{cases} H_q(X) & q < n \\ 0 & q > n \end{cases}$

(for  $H$ -singular homology)

Induction on  $n$ :  $n=0$   $X^0 = \coprod \text{pts} \Rightarrow H_q(X^0) = 0$  for  $q > 0$

Suppose true for  $n-1$

(because of Dimension + Additivity Axioms)

$$H_{q+1}(X^n, X^{n-1}) \rightarrow H_q(X^{n-1}) \rightarrow H_q(X^n) \rightarrow H_q(X^n, X^{n-1}) \rightarrow H_{q-1}(X^{n-1})$$

If  $q \neq n-1, n$ , the outer groups are 0, so

$$H_q(X^{n-1}) \cong H_q(X^n)$$

$$\text{Thus, for } q > n \quad H_q(X^n) = H_q(X^{n-1}) = H_q(X^{n-2}) = \dots = H_q(X^0) = 0$$

$$\text{And for } q < n, H_q(X^n) = H_q(X^{n+1}) = H_q(X^{n+2}) = \dots =$$

This proves the claim: if  $\dim X < \infty$ , i.e.  $X = X^N$  for  $N \gg 0$ .

Lemma  $H_q(X^{n+m}) = H_q(X)$  for  $m \gg 0$ .

Proof of Lemma:  $\exists$  rep. of a  $q$  cycle in  $H_q(X)$ :

$$\sigma = \sum a_i T_i$$

$$a_i \in \mathbb{Z}, T_i: \Delta^q \rightarrow X$$

$\text{Im } T_i \subset X^{N_i}$  for some  $N_i$

$\text{Im } \sigma \subset X^N$  for some  $N$

Thus,  $[\sigma]$  comes from  $H_q(X^N)$

This gives surjectivity



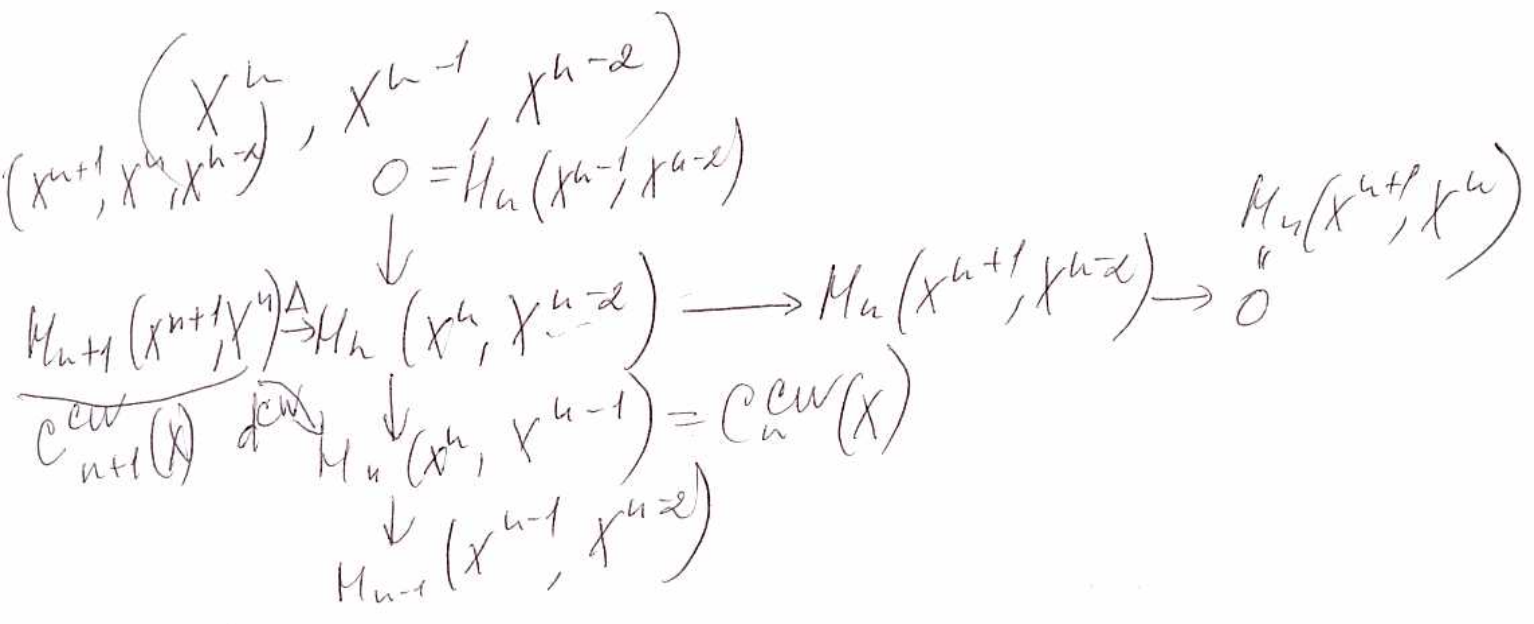
Injectivity similar

$\delta \in H_q(X^{q+n})$  which maps to 0 in  $H_q(X)$

$$\delta = d\tilde{c}, \tilde{c} = \sum_{i=1}^n a_i T_i \quad T_i: \Delta^{q+1} \rightarrow X$$

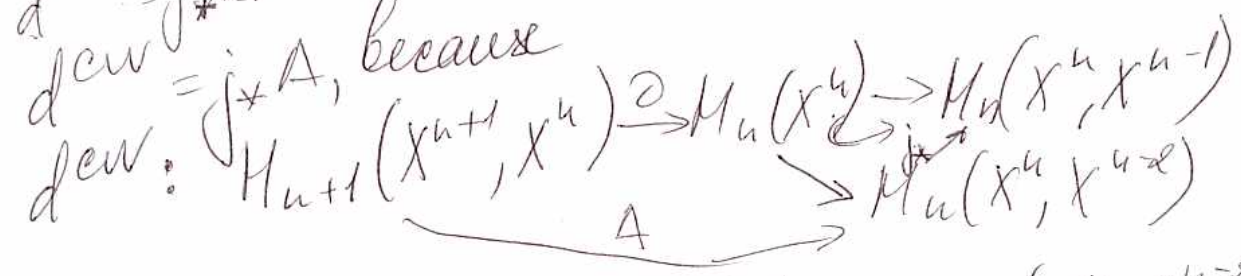
$\text{Im } \tilde{c} \subset X^N$  for  $N \gg 0$ .

$\Rightarrow \delta = d\tilde{c} = 0 \in H_q(X^N)$  Lemma proven  
Claim proven.



$$d^{CW} = j_* \Delta$$

$d^{CW} = j_* \Delta$ , because



$$\text{Then } H_n^{CW}(X) = \text{Ker } d^{CW} / \text{Im } d^{CW} = H_n(X^n, X^{n-2}) / \text{Im } \Delta =$$

$$= H_n(X^{n+1}, X^{n-2}) = H_n(X^{n+1}, X^{n-1}) = H_n(X)$$