Cellular Homology

$H_\bullet(X) = H_\bullet(X;\mathbb{Z})$ a homology theory for CW complexes X

$$H_\bullet^{cw}(X) = H_\bullet(C_\bullet^{cw}(X), d^{cw})$$

$$C_n^{cw}(X) = H_n(X^n, X^{n-1}) = \bigoplus_{\substack{n\text{-cells}\\ e^n_\alpha \text{ of } X}} \mathbb{Z} e^n_\alpha$$

$$d^{cw}(e^1_\alpha) = [1_\alpha] - [-1_\alpha]; \quad d^{cw}(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

$$d_{\alpha\beta} : S^{n-1} = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_{\beta'} S^{n-1}_{\beta'} \to S^{n-1}_\beta$$

the degree of map

Def. $k > 0$, $f: S^k \to S^k$, $H_k(S^k) \xrightarrow{f_*} H_k(S^k)$

$$\begin{array}{ccc} H_k(S^k) & \xrightarrow{f_*} & H_k(S^k) \\ \| & & \| \\ \mathbb{Z} \ni 1 & \longmapsto & \deg f \in \mathbb{Z} \end{array}$$

Theorem $H_\bullet^{cw}(X) = H_\bullet(X;\mathbb{Z})$ (for the homology theory we started up with)

Corollary: $H_\bullet^{cw}(X,A) = H_\bullet(X,A;\mathbb{Z})$

Five-Lemma with LES's

$$\begin{array}{ccccccccc} H_n^{cw}(A) & \to & H_n^{cw}(X) & \to & H_n^{cw}(X,A) & \to & H_{n-1}^{cw}(A) & \to & H_{n-1}^{cw}(X) \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to & H_{n-1}(A) & \to & H_{n-1}(X) \end{array}$$

as soon as we have natural maps between $H_\bullet^{cw}$ and H

Proof: 1. Claim $H_q(X^n) = \begin{cases} H_q(X) & q < n \\ & q > n \end{cases}$

(for H = singular homology)

Induction on n: $n = 0$ $X^0 = \coprod$ pts $\Rightarrow H_q(X^0) = 0$ for $q \neq 0$

(because of Dimension + Additivity Axiom)

Suppose true for $n-1$

$H_{q+1}(X^n, X^{n-1}) \to H_q(X^{n-1}) \to H_q(X^n) \to H_q(X^n, X^{n-1}) \to \cancel{H_{q-1}(X^n}$

If $q \neq n-1, n$, the outer groups are 0, so

$H_q(X^{n-1}) \xrightarrow{\simeq} H_q(X^n)$

Thus, for $q > n$ $H_q(X^n) = H_q(X^{n-1}) = H_q(X^{n-2}) =$
$= \dots = H_q(X^0) = 0$

And for $q < n$, $H_q(X^n) = H_q(X^{n+1}) = H_q(X^{n+2}) = \dots =$

$\cancel{\text{if}}\ H_q(X^{n+m})$ ~~for any~~ $m \geq 0$

This proves the claim ~~if~~ $\dim X < \infty$, i.e. $X = X^N$ for $N >> 0$.

$\cancel{= H_q(X^{q+1})} = \lim_{m \to \infty} H_q(X^{q+m})$

Lemma $H_q(X^{n+m}) = H_q(X)$ for $m >> 0$.

$H_q(X^{q+}$

Proof of Lemma: γ rep. of a q cycle in $H_q(X)$:

$\gamma = \sum a_i T_i$

$a_i \in \mathbb{Z}$, $T_i : \Delta^q \to X$

$\operatorname{Im} T_i \subset X^{N_i}$ for some N_i

$\operatorname{Im} \gamma \subset X^N$ for some N

Thus, $[\gamma]$ comes from $H_q(X^N)$

This gives surjectivity

Injectivity similar

$\sigma \in H_q(X^{q+n})$ wich maps to 0 in $H_q(X)$

$\sigma = d\tau$, $\tau = \sum_{i=1}^{\varepsilon_i} a_i T_i$ $\quad T_i : \Delta^{q+1} \to X$

$\operatorname{Im} \tau \subset X^N$ for $N \gg 0$.

$\Rightarrow \sigma = d\tau = 0 \in H_q(X^N)$ Lemma proven

Claim proven.

(X^n, X^{n-1}, X^{n-2})

(X^{n+1}, X^n, X^{n-2})

$$\begin{tikzcd}
 & 0 = H_n(X^{n-1}, X^{n-2}) \arrow[d] & & H_n(X^{n+1}, X^n) \arrow[d, equal] \\
\underbrace{H_{n+1}(X^{n+1}, X^n)}_{C^{cw}_{n+1}(X)} \arrow[r, "\Delta"] \arrow[dr, "d^{cw}"'] & H_n(X^n, X^{n-2}) \arrow[r] \arrow[d] & H_n(X^{n+1}, X^{n-2}) \arrow[r] & 0 \\
 & H_n(X^n, X^{n-1}) = C^{cw}_n(X) \arrow[d] & & \\
 & H_{n-1}(X^{n-1}, X^{n-2}) & &
\end{tikzcd}$$

$d^{cw} = j_* \Delta$

$d^{cw} = j_* \Delta$, because

$$\begin{tikzcd}
d^{cw} : H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[rr, bend right, "\Delta"'] & H_n(X^n) \arrow[r] \arrow[dr] & H_n(X^n, X^{n-1}) \\
 & & H_n(X^n, X^{n-2}) \arrow[u, "j_*"']
\end{tikzcd}$$

Then $H^{cw}_n(X) = \operatorname{Ker} d^{cw} / \operatorname{Im} d^{cw} = H_n(X^n, X^{n-2}) / \operatorname{Im} \Delta =$

$= H_n(X^{n+1}, X^{n-2})$

So, $H^{cw}_n(X) = H_n(X^{n+1}, X^{n-2}) = H_n(X^{n+1}) = H_n(X)$

$\underset{\substack{\| \\ 0}}{H_n(X^{n-2})} \to H_n(X^{n+1}) \xrightarrow{\sim} H_n(\cancel{X}^{n+1}, X^{n-2})$