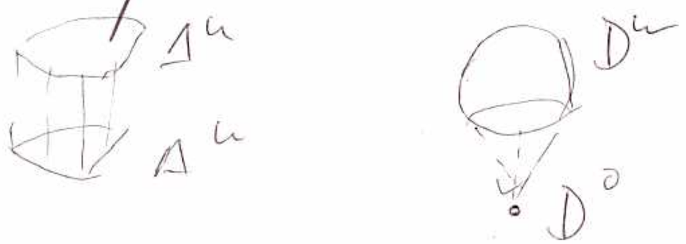
Math 8306 Lecture 21 (M 10.25.2004)

Cellular Homology

$S^n, \mathbb{C}P^n, \mathbb{R}P^n$ Δ-complex structures are more complicated than CW structures

S^n



Whant cellular homology.
However, start with a homology theory $H_\bullet(X, A; \mathbb{Z})$ for CW pairs

$$H_\bullet(X,A; \mathbb{Z}) = H_\bullet(X,A)$$

$\forall X$ CW cx, define $(C_\bullet^{cw}(X), d = d^{cw})$ as

$C_n^{cw}(X) := H_n(X^n, X^{n-1})$

X^k = k-skeleton of X

$$\begin{tikzcd}
d = d^{cw} : C_n^{cw}(X) \arrow[r] \arrow[d, equal] & C_{n-1}^{cw}(X) \arrow[dr, equal] & \\
H_n(X^n, X^{n-1}) \arrow[drr, "\partial"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & H_{n-1}(X^{n-1}) \arrow[u, "j_*"']
\end{tikzcd}$$

$j : (X^{n-1}, \emptyset) \to (X^{n-1}, X^{n-2})$

$d^2 = 0$, because

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[r, "d^{cw}"] \arrow[dr, "\partial"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "d^{cw}"] \arrow[dr, "\partial"'] & H_{n-2}(X^{n-2}, X^{n-3}) \\
& H_{n-1}(X^{n-1}) \arrow[u, "j_*"'] & H_{n-2}(X^{n-2}) \arrow[u, "j_*"']
\end{tikzcd}$$

$\partial j_* = 0$ as 2 subsequent maps in a LES

Thus, $(C_\bullet^{cw}(X), d^{cw})$ is a complex, called the <u>cellular complex</u>, and its homology $H_\bullet^{cw}(X) = \operatorname{Ker} d^{cw} / \operatorname{Im} d^{cw}$ is called <u>cellular homology</u>

<u>Prop.</u> 1. $H_n(X^n, X^{n-1}) = C_n^{cw}(X) \cong \bigoplus_{\substack{\text{the } n\text{-cells} \\ \{e^n_\alpha\} \text{ of } X}} \mathbb{Z} = $

$\cong \bigoplus \mathbb{Z} e^n_\alpha$ where e^n_α ~~are~~ $\oplus \to H_n(X^n, X^{n-1})$ generators of $H_n(D^n_\alpha, \partial D^n_\alpha) \cong \mathbb{Z}$

2. $H_q(X^n, X^{n-1}) = 0$ for $q \neq n$ ~~and $n \geq 1$~~ and $n \geq 0$.

<u>Proof</u>: B. $H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n / X^{n-1}) \cong \tilde{H}_q(\bigvee_\alpha S^n)$

$\cong \bigoplus H_q(\coprod_\alpha D^n, \coprod_\alpha \partial D^n)$

isomorphic

$\tilde{H}_q(S^n) = \tilde{H}_q(D^n / \partial D^n) = H_q(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z}, & q = n \\ 0 & q \neq n \end{cases}$

$$\begin{tikzcd} e^n_\alpha \in H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\sim"] \arrow[d, "(\varphi_\alpha)_*"] & \tilde{H}_n(D^n_\alpha / \partial D^n_\alpha) \arrow[d, "(\varphi_\alpha)_*"] \\ H_n(X^n, X^{n-1}) \arrow[r, "\sim"] & \tilde{H}_n(X^n / X^{n-1}) \end{tikzcd}$$

Prop.: $d^{cw}(e^n_\alpha) = \sum_{\beta \in \{(n-1)\text{-cells of } X\}} d_{\alpha\beta} e^{n-1}_\beta$ $\quad n \geq 2$ or $n = 0$.

Where $d_{\alpha\beta} \in \mathbb{Z}$ are the "degrees" of the following maps

$$S^{n-1}_\alpha = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \longrightarrow S^{n-1}_\beta$$

$$X^{n-1}/X^{n-2} = \bigvee_{\beta' \in \{\text{all } (n-1)\text{cells}\}} S^{n-1}$$

collapse all the $(n-1)$ spheres but S^{n-1}_β to a point.

The degree of $d_{\alpha\beta}$ is defined:

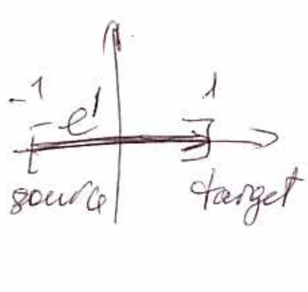
$$\begin{tikzcd} H_{n-1}(S^{n-1}_\alpha) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(S^{n-1}_\beta) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[d, ni] & \mathbb{Z} \arrow[d, ni] \\ 1 \arrow[r, mapsto] & d_{\alpha\beta} \times 1 \end{tikzcd}$$

chose some isomorphism

For $n = 1$: $d^{cw}(e^1_\alpha) = t(e^1_\alpha) - s(e^1_\alpha)$

target source

0-cells in X.



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<u>Proof:</u> $\cancel{e_\alpha^n \in H_{n-1}(S_\alpha^{n-1})}$ $\cancel{\overset{\partial D_\alpha^n}{\|}\ \ \cong \partial^{-1}}$

projection on β su

$$\begin{tikzcd}
e_\alpha^n \in H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial"] \arrow[d, "(\varphi_\alpha)_*"'] & H_{n-1}(\partial D_\alpha^n) \arrow[r, "\sigma_{\alpha\beta}"] \arrow[d] & H_{n-1}(D^n_\beta / \partial D^n_\beta) \\
e_\alpha^n \in H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[rr, bend right, "d_{cw}"'] & H_{n-1}(X^{n-1}) \arrow[r, "j_*"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[u]
\end{tikzcd}$$

$$\| \quad \bigoplus_\beta H_{n-1}(D^n_\beta, \ldots)$$

$$\sum_\beta p_\beta = id \qquad d^{cw}(e_\alpha^n) = \sum_\beta p_\beta (d^{cw} e_\alpha^n)$$

$$= \sum_\beta (\sigma_{\alpha\beta})_* e_\beta^{n-1} =$$

$$= \sum_\beta d_{\alpha\beta} e_\beta^{n-1} \quad \blacksquare$$