Math 8306 Lecture 17 (F 10.15.04)

<u>Excision Axiom</u>

Th. $C_\cdot^{\mathcal{W}}(X) \hookrightarrow C_\cdot(X)$
is a homotopy equivalence,
where $\mathcal{W}$ is an open covering of X
(constructed a homotopy inverse φ:
$C_\cdot(X) \to C_\cdot^{\mathcal{W}}(X)$
$id - i\varphi = dH + Hd$
$\varphi i = id$
)

<u>Corollary</u>: $C_\cdot^{\mathcal{W}}(X;G) := C_\cdot^{\mathcal{W}}(X) \otimes G \hookrightarrow$
$C_\cdot(X;G)$
~~i.e. is a~~ is a homotopy equivalence
($\varphi \otimes id_G$ will be a homotopy inverse).

<u>Corollary</u>: $H_\cdot(C_\cdot^{\mathcal{W}}(X;G)) \cong H_\cdot(X;G)$

~~E~~
$X = A \cup B$, A, B - open
<u>Theorem</u>: the excision homomorphism
$H_\cdot(A, A\cap B; G) \to H_\cdot(X, B; G)$
is an isomorphism

<u>Proof</u>: $\mathcal{W} = \{A, B\}$ open covering of X

$H_\cdot(X, B; G)$ is the homology of $C_\cdot(X; G)/C_\cdot(B; G) =$

$= \{\sum_i T_i \otimes g_i \mid T_i : \Delta^n \to X\} / \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to B\}$

simplecies

$= \{\sum_i T_i \otimes g_i \mid T_i : \Delta^n \to X,\ \operatorname{Im} T_i \not\subset B\}.$

$C_\cdot^{\mathcal{W}}(X; G)/C_\cdot(B, G) = \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to X, \operatorname{Im} T_i \subset A\}, \operatorname{Im} T_i \not\subset A \cap B\}$
$g_i \in G$

$$\begin{tikzcd}
C_\cdot(B; G) \arrow[r, hook] \arrow[d, equal] & C_\cdot(X; G) \\
C_\cdot(B; G) \arrow[r, hook] & C_\cdot^{\mathcal{W}}(X; G) \arrow[u, "i"]
\end{tikzcd}$$

i induces a homotopy equivalence on

$C_\cdot^{\mathcal{W}}(X; G)/C_\cdot(B; G) \xrightarrow{i} C_\cdot(X; G)/C_\cdot(B; G)$

homology $= H_\cdot(X, B; G)$

$= \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to A,\ g_i \in G\}, \operatorname{Im} T_i \not\subset A \cap B\}$

$= C_\cdot(A; G)/C_\cdot(A \cap B; G) \xrightarrow{\text{homology}} H_\cdot(A, A \cap B; G)$

$\Rightarrow$ get an iso of the homology groups, given by $A \hookrightarrow X$. $\blacksquare$

Δ-complexes

Δ-complexes are built inductively out of simplices Δ^n ($\cong D^n$):

X^0 = a set (a discrete top. space), $\sigma_x : \Delta^0 \to X^0$
one point space $\to \{*\} \mapsto x$ $\forall x \in X^0$

$\vdots$

X^{n-1} with a collection of characteristic maps $\sigma_\alpha : \Delta^{n-1} \to X^{n-1}$, $\alpha \in J$

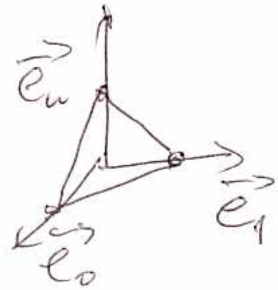
$X^n = X^{n-1} \cup_\sigma \coprod_{\beta \in K} \Delta_\beta^n$, where $\sigma_\beta : \partial \Delta_\beta^n \to X^{n-1}$

$\coprod \Delta_\beta^n \xrightarrow{\coprod \sigma_\beta}$ topological union (pushout) union such that for each β and each phase of Δ^n_β $\exists \alpha$ so that

$\sigma|_{\text{face of } \Delta^n_\beta} = \sigma_\alpha : \Delta^{n-1} \to X^{n-1}$

where the face of Δ^n_β is identified with Δ^{n-1} via a canonical linear homeomorphism.

$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}$
$= [\vec{e}_0, \vec{e}_1, \dots, \vec{e}_n]$ $\sim$ convex hull



$\forall$ collection of $n+1$ points in $\mathbb{R}^N$ in general position (not lie in a $n-1$ dim plane)

$\vec v_0, \dots, \vec v_n$ we can take the simplex $[\vec v_0, \dots, \vec v_n]$ Then $\exists$ canonical linear homeomorphism $[\vec v_0, \dots, \vec v_n] \to \Delta^n$, taking $\vec v_i$ to $\vec e_i$.

Ordering on the set of vertices of a face of Δ^n is taken to be the one induced from the ordering $0, 1, \dots, n$

$\partial_i : \Delta^{n-1} \to \Delta^n$

$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.

$[\vec e_0, \dots, \vec e_{n-1}] \mapsto [\vec e_0, \dots, \vec e_{i-1}, \vec e_{i+1}, \dots, \vec e_n]$

$\Delta^n(X) := \{\sum n_\beta \sigma_\beta \mid n_\beta \in \mathbb{Z}, \sigma_\beta : \Delta^n_\beta \to X$ characteristic map$\}$

if X is a Δ-complex

$d_i \sigma_\beta = \sigma_\beta |_{[\vec e_0, \dots, \hat{\vec e_i}, \dots, \vec e_n]}$ is one of the σ_α's

$d\sigma_\beta = \sum (-1)^i d_i \sigma_\beta$ the boundary of σ_β | $d^2 = 0$

$(\Delta_\bullet(X), d)$ is a complex. Its $H_\bullet^\Delta(X) := \mathrm{Ker}\, d / \mathrm{Im}\, d$

simplicial homology of X