Math 8306 Lecture 17 (F10.15.04)

Excision Axiom

Th. $C_\bullet^{\mathcal{W}}(X) \hookrightarrow C_\bullet(X)$

is a homotopy equivalence,
where $\mathcal{W}$ is an open covering of X
(constructed a homotopy inverse φ:
$C_\bullet(X) \to C_\bullet^{\mathcal{W}}(X)$
$id - i\varphi = dH + Hd$
$\varphi i = id$
)

Corollary: $C_\bullet^{\mathcal{W}}(X;G) := C_\bullet^{\mathcal{W}}(X) \otimes G \hookrightarrow C_\bullet(X;G)$

~~i.e. is a~~ is a homotopy equivalence
($\varphi \otimes id_G$ will be a homotopy inverse).

Corollary: $H_\bullet(C_\bullet^{\mathcal{W}}(X;G)) \cong H_\bullet(X;G)$

$X = A \cup B$, A, B - open

Theorem: the excision homomorphism
$H_\bullet(A, A \cap B; G) \to H_\bullet(X, B; G)$
is an isomorphism

<u>Proof</u>: $\mathcal{W} = \{A, B\}$ open covering of X

$H_\bullet(X,B;G)$ is the homology of $C_\bullet(X;G)/C_\bullet(B;G) =$

$$= \{\sum_i \underbrace{T_i}_{\text{simplices}} \otimes g_i \mid T_i : \Delta^n \to X\} / \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to B\}$$

$$= \{\sum_i T_i \otimes g_i \mid T_i : \Delta^n \to X, \ \operatorname{Im} T_i \not\subset B\}.$$

$$C_\bullet^{\mathcal{W}}(X;G)/C_\bullet(B,G) = \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to X, \ \operatorname{Im} T_i \subset A \ \&, \ \operatorname{Im} T_i \not\subset A\cap B\}$$
$g_i \in G$

$$\begin{array}{ccc} C_\bullet(B;G) & \hookrightarrow & C_\bullet(X;G) \\ \| & & \uparrow i \\ C_\bullet(B;G) & \hookrightarrow & C_\bullet^{\mathcal{W}}(X;G) \end{array}$$

i induces a homotopy equivalence on

$$C_\bullet^{\mathcal{W}}(X;G)/C_\bullet(B;G) \xrightarrow{i} C_\bullet(X;G)/C_\bullet(B;G)$$

homology $= H_\bullet(X,B;G)$

$$= \{\sum T_i \otimes g_i \mid T_i : \Delta^n \to A, \ g_i \in G \ \&, \ \operatorname{Im} T_i \not\subset A\cap B\}$$

$$= C_\bullet(A;G)/C_\bullet(A\cap B; G) \xrightarrow{\text{homology}} H_\bullet(A, A\cap B; G)$$

$\Rightarrow$ get an iso of the homology groups, given by $A \hookrightarrow X$. ∎

Δ-complexes

Δ-complexes are built inductively out of simplices Δ^n ($\cong D^n$):

X^0 = a set (a discrete top. space), $\sigma_x : \Delta^0 \to X^0$
one point space $\{*\} \mapsto x$ $\forall x \in X^0$

⋮

X^{n-1} with a collection of characteristic maps $\sigma_\alpha : \Delta^{n-1} \to X^{n-1}$, $\alpha \in J$

$X^n = X^{n-1} \cup_\sigma \coprod_{\beta \in K} \Delta_\beta^n$, where $\sigma_\beta : \partial \Delta_\beta^n \to X^{n-1}$

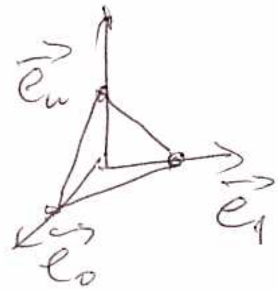
$\coprod \Delta_\beta^n \xrightarrow{\coprod \sigma_\beta}$ topological union (pushout) union

such that for each β and each face of Δ^n_β $\exists \alpha$ so that

$\sigma|_{\text{face of } \Delta^n_\beta} = \sigma_\alpha : \Delta^{n-1} \to X^{n-1}$

where the face of Δ^n_β is identified with Δ^{n-1} via a canonical linear homeomorphism.

$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}$
$= [\vec{e}_0, \vec{e}_1, \dots, \vec{e}_n]$ ← convex hull



$\forall$ collection of $n+1$ points in $\mathbb{R}^N$ in general position (not lie in a $n-1$ dim plane)

$\vec{v_0}, \dots, \vec{v_n}$ we can take the simplex $[\vec{v_0}, \dots, \vec{v_n}]$ Then $\exists$ canonical linear homeomorphism $[\vec{v_0}, \dots, \vec{v_n}] \to \Delta^n$, taking $\vec{v_i}$ to $\vec{e_i}$.

Ordering on the set of vertices of a face of Δ^n is taken to be the one induced from the ordering $0, 1, \dots, n$

$\partial_i : \Delta^{n-1} \to \Delta^n$

$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.

$[\vec{e_0}, \dots, \vec{e_{n-1}}] \mapsto [\vec{e_0}, \dots, \vec{e_{i-1}}, \vec{e_{i+1}}, \dots, \vec{e_n}]$

$\Delta^n(X) := \{ \sum n_\beta \sigma_\beta \mid n_\beta \in \mathbb{Z}, \underbrace{\sigma_\beta : \Delta^n_\beta \to X}_{\text{characteristic map}} \}$

if X is a Δ-complex

$d_i \sigma_\beta = \sigma_\beta |_{[\vec{e_0}, \dots, \hat{\vec{e_i}}, \dots, \vec{e_n}]}$ is one of the σ_α's

$d \sigma_\beta = \sum (-1)^i d_i \sigma_\beta$ the boundary of σ_β | $d^2 = 0$

$(\Delta_\bullet(X), d)$ is a complex. Its $H_\bullet^\Delta(X) := \operatorname{Ker} d / \operatorname{Im} d$

<u>simplicial homology of X</u>