Math 8306 Lecture 16 (W 10.13.2004)

<u>Excision Axiom for Singular Homology</u>

$\mathcal{W}$ open covering of X

$C_n^{\mathcal{W}}(X) = \langle T : \Delta^n \to X \mid \operatorname{Im} T \subset W$ for some $W \in \mathcal{W} \rangle$

$\overset{i}{\hookrightarrow} C_n(X)$

$d\, C_n^{\mathcal{W}} \subset C_{n-1}^{\mathcal{W}}$

<u>Theorem</u> $C_\cdot^{\mathcal{W}}(X) \overset{i}{\hookrightarrow} C_\cdot(X)$ is a h. equivalence

<u>Proof</u>: need $\varphi : C_\cdot(X) \to C_\cdot^{\mathcal{W}}(X)$, chain map,

$i\varphi \sim id$, $\varphi i \sim id$

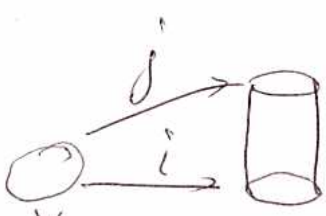
We'll construct $H : C_\cdot(X) \to C_{\cdot+1}(X)$ and define φ from the following equation:

$id - i\varphi = Hd + dH$, show $\varphi i = id$.

We had: $sd_X : C_n(X) \to C_n(X)$ chain map

$sd_X \sim id$ by "acyclic models"

(like in proving $i_* \sim j_*$ for used to prove Homotopy axiom).



X

see Selick.

We can find a chain homotopy

$h_X : C_n(X) \to C_{n+1}(X)$

$dh_X + h_X d = id - sd_X$ (construct $h_{\Delta^n}(id_{\Delta^n})$, then

$$\text{define } h_X(T) = T_* h_{\Delta^n}(id_{\Delta^n})$$

$$\begin{array}{l} T: \Delta^n \to X \\ id \uparrow \qquad T = T_*(id_{\Delta^n}) \\ \Delta^n \end{array}$$

<u>Note</u>: $\operatorname{Im} h(T) \subset \operatorname{Im} T$

$\operatorname{Im} sd(T) \subset \operatorname{Im} T$

$\operatorname{Im} sd^p(T) \subset \operatorname{Im} sd^q(T)$, if $p \geq q$

Construct a homotopy h_m $sd_X^m \sim id$ $\forall m \geq 0$.

$$h_m := \sum_{i=0}^{m-1} h \circ sd^i \quad \left(\begin{array}{l}\text{for } m=0, \; h_m = 0 \\ m = 1, \; h_m = h\end{array}\right)$$

$$h_m : C_n(X) \to C_{n+1}(X)$$

Check that it's a (chain) homotopy:

$$dh_m + h_m d = \sum_{i=0}^{m-1} \left(dh\, sd^i + h\, sd^i d\right)$$

you can interchage d with sd^q because sd is a chain ~~hom.~~ map

$$= \underbrace{(dh + hd)}_{id - sd} \sum_{i=0}^{m-1} sd^i = (id - sd) \sum_{i=0}^{m-1} sd^i$$

$$= id - sd^m$$

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Define $M(T) := h_{m(T)}(T)$, where

$T : \Delta^n \to X \qquad m(T) = \min\{m \mid sd^m T \in C_n^{\mathcal{W}}(X)\}$

$dM(T) + MdT =$

$= dh_{m(T)}(T) + h_{m(T)}dT + MdT - h_{m(T)}dT$

$= T - (sd^{m(T)}(T) - MdT + h_{m(T)}dT)$

Set $\varphi(T) := sd^{m(T)}(T) - MdT + h_{m(T)}dT$

Need to check $\varphi : C_n(X) \to C_n^{\mathcal{W}}(X)$

φ is a chain map $\quad \varphi i = id$

Checking : $sd^{m(T)}(T) \in C_n^{\mathcal{W}}(X)$

$$h_{m(T)}dT - MdT = \sum_{i=0}^{m(T)-1} h\, sd^i dT - \sum_{j=0}^{n} (-1)^j M(d_jT)$$

d_jT: jth face of T

$$= \sum_{i=0}^{m(T)-1} h\, sd^i \sum_{j=0}^{n} (-1)^j d_jT - \sum_{j=0}^{n} (-1)^j \cdot \sum_{i=0}^{m(d_jT)-1} h \circ sd^i d_jT =$$

$$= \sum_{j=0}^{n} (-1)^j \left(\sum_{i=0}^{m(T)-1} h\, sd^i d_jT - \sum_{i=0}^{m(d_jT)-1} h\, sd^i d_jT \right)$$

$m(d_jT) \le m(T)$

$$= \sum_{j=0}^{n} (-1)^j \sum_{i=m(d_j T)}^{m(T)-1} h\, sd^i d_j T$$

$$\mathrm{Im}\, sd^i d_j T \subset \mathrm{Im}\, sd^{m(d_j T)} d_j T$$

c elements of $\mathcal{W}$

$$sd^i d_j T \in C_{n-1}^{\mathcal{W}}(X)$$

$\mathrm{Im}\, h\alpha \subset \mathrm{Im}\, \alpha \Rightarrow$ $h\, sd^{i} d_j T \in C_n^{\mathcal{W}}(X)$

$$\underbrace{d\varphi T - \varphi d T}_{} = d sd^{m(T)}(T) - dHdT + d h_m(T) \cancel{T} - \sum_{j=0}^{n} (-1) \ldots\ldots$$

$\Uparrow$

$$T - \varphi(T) = dHT + HdT$$

$\varphi i \overset{?}{=} id \quad \varphi i(T) = \varphi(T)$

$$T \in C_n^{\mathcal{W}}(X) \overset{i}{\subset} C_n(X)$$

$\cancel{E}$ $\mathrm{Im}(T) = 0 \Rightarrow H = h_0 = 0$

in $C_i^{\mathcal{W}}$