

Existence Axiom for Singular Homology

W open covering of X

$$C_n^w(X) = \langle T: \Delta^n \rightarrow X \mid \text{Im } T \subset W \text{ for some } W \in \mathcal{W} \rangle$$

$$\hookrightarrow C_n(X)$$

$$d C_n^w \subset C_{n-1}^w$$

Theorem $C_n^w(X) \xrightarrow{i} C_n(X)$ is a h. equivalence

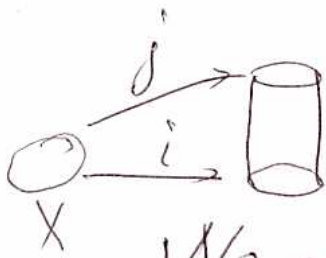
Proof: need $\psi: C_n(X) \rightarrow C_n^w(X)$, chain map,
 $\psi \circ i \sim \text{id}$, $i \circ \psi \sim \text{id}$

We'll construct $H: C_n(X) \rightarrow C_{n+1}^w(X)$ and define ψ from the following equation:

$$\text{id} - i\psi = Hd + dH, \text{ show } \psi i = \text{id}.$$

We had: $sdx: C_n(X) \rightarrow C_n(X)$ chain map
 $sdx \sim \text{id}$ by "acyclic models"

(like in proving $i_* \simeq j_*$ for used to prove Homotopy axiom).
 see Selick.



We can find a chain homotopy $h_x: C_n(X) \rightarrow C_{n+1}^w(X)$

$$dh_x + h_x d = \text{id} - sdx \text{ (construct } h_{\Delta^n}(\text{id}_{\Delta^n}), \text{ then$$

define $h_x(T) = T_*^{16} h_{\Delta^n}(id_{\Delta^n})$

$$T: \Delta^n \rightarrow X$$

$$\begin{array}{c} id \uparrow \\ \Delta^n \end{array} \quad T = T_*(id_{\Delta^n})$$

Note: $Im h(T) \subset Im T$
 $Im sd(T) \subset Im T$

$Im sd^p(T) \subset Im sd^q(T)$, if $p \geq q$

Construct a homotopy $h_m: sd^m X \rightarrow id \forall m \geq 0$.

$$h_m := \sum_{i=0}^{m-1} h \circ sd^i \quad \left(\begin{array}{l} \text{for } m=0, h_m = 0 \\ m=1, h_m = h \end{array} \right)$$

$$h_m: C_n(X) \rightarrow C_{n+1}(X)$$

Check that it's a (chain) homotopy:

$$dh_m + h_m d = \sum_{i=0}^{m-1} (dh \circ sd^i + h \circ sd^i \circ d)$$

you can interchange d with sd^i because sd is a chain map.

$$= \underbrace{(dh + hd)}_{id - sd} \sum_{i=0}^{m-1} sd^i = (id - sd) \sum_{i=0}^{m-1} sd^i = id - sd^m$$

Define $M(T) := h_{m(T)}(T)$, where

$$T: A^n \rightarrow X \quad m(T) = \min \{m \mid sd^m T \in C_n^w(X)\}$$

$$\begin{aligned} dM(T) + MdT &= \\ &= d h_{m(T)}(T) + h_{m(T)} dT + MdT - h_{m(T)} dT \\ &= T - \left(sd^{m(T)}(T) \mp MdT + h_{m(T)} dT \right) \end{aligned}$$

$$\text{Set } \varphi(T) := sd^{m(T)}(T) - MdT + h_{m(T)} dT$$

Need to check $\varphi: C_n(X) \rightarrow C_n^w(X)$

φ is a chain map $\varphi \circ i = id$

Checking: $sd^{m(T)}(T) \in C_n^w(X)$

$$h_{m(T)} dT - MdT = \sum_{i=0}^{m(T)-1} h sd^i dT -$$

$$- \sum_{j=0}^n (-1)^j M(d_j T)$$

$$= \sum_{i=0}^{m(T)-1} h sd^i \sum_{j=0}^n (-1)^j d_j T - \sum_{j=0}^n (-1)^j \cdot \sum_{i=0}^{m(d_j T)-1} h sd^i d_j T =$$

$$= \sum_{j=0}^n (-1)^j \left(\sum_{i=0}^{m(T)-1} h sd^i d_j T - \sum_{i=0}^{m(d_j T)-1} h sd^i d_j T \right)$$

$$m(d_j T) \leq m(T)$$

$$= \sum_{j=0}^n (-1)^j \sum_{i=m(d_j \cdot T)}^{m(T)-1} h s d^i d_j \cdot T$$

$\text{Im } s d^i d_j \cdot T \subset \text{Im } s d^{m(d_j \cdot T)} d_j \cdot T$
 elements of \mathcal{W}

$$\text{Im } h d \subset \text{Im } d \Rightarrow \begin{matrix} s d^i d_j \cdot T \in C_u^w(X) \\ h s d^i d_j \cdot T \in C_u^w(X) \end{matrix}$$

$$d\psi T - \psi dT = d s d^{m(T)}(T) - d M d T + d h m(T) \cdot T$$

$$\underbrace{\hspace{10em}}_{\uparrow \uparrow} = \sum_{j=0}^n (-1)^j \dots$$

$$T - \psi(T) = d M T + M d T$$

$$\psi_i = id \quad \psi_i(T) = \psi(\psi)$$

$$T \in C_u^w(X) \subset C_u(X)$$

$$\mathbb{E} \text{Im}(T) = 0 \Rightarrow M = h_0 = 0$$

in $C_u^w(X)$