Math 8306 Lecture 15 (M 10.11. 2004)

Excision Axiom for Singular Homology

$\mathcal{W}$ open covering of X: $X = \bigcup_{W \in \mathcal{W}} W$, W open in X

subcomplex $C_n^{\mathcal{W}}(X) := \langle T : \Delta^n \to X \mid \operatorname{Im} T \subset W$ for some $W \in \mathcal{W} \rangle$

$\subset \overset{\text{abelian group}}{C_n(X)}$ $= \{ \sum n_i T_i \mid \operatorname{Im} T_i \subset W \}$

$C_n^{\mathcal{W}}$ is a subgroup (and actually subcomplex) by construction

$d\, C_n^{\mathcal{W}}(X) \subset C_{n-1}^{\mathcal{W}}(X)$

$\Rightarrow C_\bullet^{\mathcal{W}}(X)$ is a subcomplex of $C_\bullet(X)$

$$C_\bullet^{\mathcal{W}}(X;G) := C_\bullet^{\mathcal{W}}(X) \otimes G \hookrightarrow \overset{\text{subcomplex}}{C_\bullet(X;G)}$$

$$\| \\ \{ \sum T_i \otimes g_i \mid \operatorname{Im} T_i \subset W \text{ for some } W \text{ in } \mathcal{W} \}$$

<u>Theorem</u>: $C_\bullet^{\mathcal{W}}(X) \overset{i}{\hookrightarrow} C_\bullet(X)$ is a (chain) homotopy equivalence

<u>Proof</u>: Let's call the inclusion "i"

Let's define the homotopy inverse:

$$\varphi : C_\bullet(X) \to C_\bullet^{\mathcal{W}}(X)$$

free abelian group generated by ...

$\varphi(T) := sd_X^{m(T)}(T)$ baricentric subdivision to the power $m(T)$

$T : \Delta^n \to X$

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$m(T) := \min\{m \mid sd_X^m(T) \in C_n^{\mathcal{W}}(X)\}$

Applying the Lebeg's lemma we see that it exists.

$d\varphi \overset{?}{=} \varphi d$

$d\varphi(T) = d\,(sd_X)^{m(T)}(T)$

$= (sd_X)^{m(T)}(dT) = \sum_i (sd_X)^{m(T)}(d_iT_i)(-1)^i$

$dT = \sum_{i=0}^{n} d_iT_i(-1)^i$

$\overset{?}{\neq} \sum_i (-1)^i (sd_X)^{m(d_iT)}(d_iT) = \varphi d T$

$\varphi i = id$

$i\varphi(T) = sd_X^{m(T)}(T) \sim \cancel{id}$

$sd_X^{m(T)}(T) - T = s_{m(T)}dT + d\,s_{m(T)}T$

$i\varphi(T) = sd_X^{m(T)}(T) = T - s_{m(T)}dT - ds_{m(T)}T$

$i\varphi(\sum n_iT_i) = \sum n_i sd^{m(T_i)}(T_i)$

$= \sum n_i T_i - \sum n_i s_{m(T_i)} dT_i -$

$- \sum n_i d s_{m(T_i)} T_i)$

$\sum n_i T_i \in \operatorname{Ker} d$ $\quad \sum_i s_{m(T_i)}\, d(n_iT_i) \overset{?}{\Downarrow} \in \operatorname{Im} d$

$\quad \to d \sum n_i s_m(T_i)(T_i)$

$$sd_x^{m(T)}(T) - T = s_{m(T)} d T + d s_{m(T)} T$$

apply sd_x one more time

$$sd_x^{m(T)+1}(T) - \cancel{sd_x} T = (s_{m(T)+1} dT) +$$
$$+ d s_{m(T)+1} T$$

$$\sum n_i T_i \in \ker d \Rightarrow \sum_i s_m(T_i) d(n_i T_i) \in \operatorname{Im} d.$$

$$\Rightarrow i\varphi\left(\sum n_i T_i\right) = \sum n_i T_i$$

in $H_\bullet(C_\bullet(X)) = H_\bullet(X;\mathbb{Z})$

$$\Rightarrow H_\bullet^{\#}(C_\bullet^{W}(X)) \xrightarrow{i_*} H_\bullet(C_\bullet(X))$$