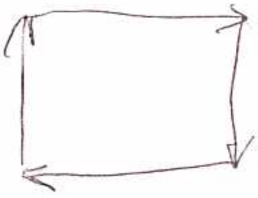
Math 8306 Lecture 14 (F10.08.2004)

2.3.1 (from Hatcher)

$$\mathbb{RP}^2 = S^2/\{x \sim -x\} = \{\text{lines in } \mathbb{R}^3\}$$



$\cong$ cellular or simplicial homology $\Longrightarrow$

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$S^1 \cong \mathbb{RP}^1 \subset \mathbb{RP}^2$$

$\mathbb{RP}^1 = \{\text{lines in the } xy \text{ plane}\}$

 $\cong \mathbb{C}/\sim \; =$ 

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

LES for $(\mathbb{RP}^2, S^1)$

$$\cdots \to \underset{H_2(\mathbb{RP}^2)}{0} \to \underset{H_2(\mathbb{RP}^2, S^1)}{\mathbb{Z}} \to \underset{H_1(S^1)}{\mathbb{Z}} \to \underset{H_1(\mathbb{RP}^2)}{\mathbb{Z}_2} \to \underset{H_1(S^2)}{0}$$

$H_2(\mathbb{RP}^2, S^1) = H_2(\mathbb{RP}^2/S^1) = \tilde{H}_2(S^2) = H_2(S^2) = \mathbb{Z}$

S^1

 $/S_1 =$ 

14

$$h_n(X,A) = \mathrm{Tor}\, H_n(X,A)$$ ← not exact

$$0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0$$

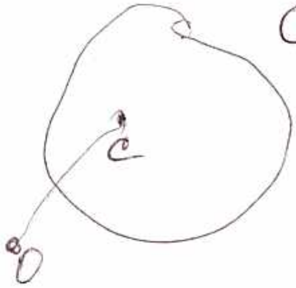
$$h_n'(X,A) = H_n(X,A) / \mathrm{Tor}\, H_n(X,A)$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\pm 2} \mathbb{Z} \to 0 \to 0$$

not exact here

Excision for Singular Homology

Theorem: $B, A \subset X$ top pairs $X = A \cup B$ and A and B are open in X.
Then $\exists$ natural isomorphism
$$H_\bullet(A, A\cap B) \xrightarrow{\simeq} H_\bullet(X,B)$$
In fact, it's given by $A \hookrightarrow X$.



$C \subset \mathbb{R}^N$
convex

$$S_{n,c} : C_n(C) \to C_{n+1}(C)$$

$$T : \Delta^n \to C$$

$$S_{n,c}\big((1-t)z + t\vec{e}_{n+1}\big) := (1-t)\,T(z) + t\,c.$$

$$\Delta^{n+1} = \{(1-t)z + t\vec{e}_{n+1} \mid t \in I,\ z \in \Delta^n \subset \mathbb{R}^{n+1}\}$$

$$\vec{e}_{n+1} = (0,\dots,0,1)$$

$\mathbb{R}^{n+2} = \{(x_0, x_1, \dots, x_{n+1})\} \supset \mathbb{R}^{n+1} = \{(x_0, \dots, x_n)\}$

Barycentric Subdivision

$sd_{\Delta^n}(id_{\Delta^n}) := (-1)^n s_{n,\sigma}\, sd_{\Delta^n}(d(id_{\Delta^n}))$

recursive definition of $sd_{\Delta^{n-1}}$

$\Delta^n \xrightarrow{id} \Delta^n \quad sd_{\Delta^n}: C_n(X) \to C_n(X) \quad sd_{\Delta^0} = id$

$sd_{\Delta^0}(id_{\Delta^0}) = id_{\Delta^0}$

where σ is the barycenter of $\Delta^n = \{(t_0, t_1, t_n) \mid \sum t_i = 1,\ t_i \geq 0\}$

$\sigma = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right)$

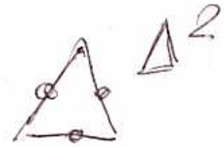
$d(\overset{0}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{1}{\bullet}) = 1 - 0$

$1\ 0 : \Delta^0 \to \Delta^1 = I$

1

$0 \longleftarrow \bullet\ 1/2 \quad \overset{1/2}{\bullet} \longrightarrow 1$

$sd_{\Delta^1}(id_{\Delta^1}) = -\overset{0}{\bullet}\!\!\longrightarrow^{1/2} + \dots$



$sd_X(T) = T_*(sd_{\Delta^n}(id_{\Delta^n}))$

arbitrary

$T: \Delta^n \to X$

$T = T_*(id_{\Delta^n})$ $\quad \Delta^n \xrightarrow{id_{\Delta^n}} \Delta^n \xrightarrow{T}$ "

Thus, sd_X ~~compact~~: $C_n(X) \to C_n(X)$ is natural.

Lemma: 1. sd_X is a natural chainmap ($\partial sd_X = sd_X \partial$)

2. Every n-simplex of $sd_{\Delta^n}(id_{\Delta^n})$ is spanned by vertices $\hat{\sigma}_0, \dots, \hat{\sigma}_n$ for an strictly increasing collection of subfaces of Δ^n an $\hat{\sigma}_i$ is the barycenter of σ_i.

3. Let $\mathcal{W}$ be a collection of open subsets of X covering X, $T: \Delta^n \to X$. Then $\exists N: sd^N_X(T) =$
$= \sum_i m_i T_i$ so that each $T_i \subset W$ for some $W \in \mathcal{W}$

($\Delta^n \xrightarrow{T} X$

$\{T^{-1}(W) \mid W \in \mathcal{W}\}$ open covering of Δ^n

Lebesgue's Lemma: $\exists \varepsilon > 0$ such ~~that~~ that every subset of diameter $< \varepsilon$ is contained in one element of the covering)

4. $\forall m$ $\exists$ a neutral chain homotopy
$S_X : id \simeq (sd_X)^m$

$C_n \xrightarrow{id} C_n\ (sd_0)^n : C_n \to C_n$

Construct it similar to how we constructed a homotopy for i_* and j_* in proving Homology Axiom.

(method of acyclic models: transferred this homotopy from Δ^n by $T_*(id_{\Delta^n}) = T$, using $H_q(\Delta^n) = \begin{cases} G & q = 0 \\ 0 & q \neq 0 \end{cases}$

and knowing $id = (sd_{\Delta^n})^m_*$ in homology $H_0(\Delta^n)$.