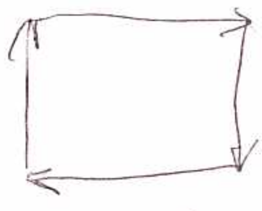
Math 8306 Lecture 14 (F10.08.2004)

2.3.1 (from Hatcher)

$$\mathbb{RP}^2 = S^2/\{x\sim -x\} = \{\text{lines in } \mathbb{R}^3\}$$



$\cong$ cellular or simplicial homology $\Longrightarrow$

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$S^1 \cong \mathbb{RP}^1 \subset \mathbb{RP}^2$$

$$\overset{\shortparallel}{\{\text{lines in the } xy \text{ plane}\}}$$

$$\cong \mathbb{C}/\!\sim\ = \bigcirc$$

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n=0,1 \\ 0 & \text{otherwise} \end{cases}$$

LES for $(\mathbb{RP}^2, S^1)$

$$\cdots \to \underset{H_2(\mathbb{RP}^2)}{0} \to \underset{H_2(\mathbb{RP}^2,S^1)}{\mathbb{Z}} \to \underset{H_1(S^1)}{\mathbb{Z}} \to \underset{H_1(\mathbb{RP}^2)}{\mathbb{Z}_2} \to \underset{H_1(S^2)}{0}$$

$$S^1 \quad = H_2(\mathbb{RP}^2/S^1) = \tilde{H}_2(S^2) = H_2(S^2) = \mathbb{Z}$$

$/S_1 =$

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$h_n(X,A) = \mathrm{Tor}\, H_n(X,A)$ ← not exact

$0 \to 0 \to 0 \to \mathbb{Z}_2 \to 0$

$h_n'(X,A) = H_n(X,A) / \mathrm{Tor}\, H_n(X,A)$

$0 \to \mathbb{Z}^{\pm 2} \to \mathbb{Z} \to 0 \to 0$

not exact here

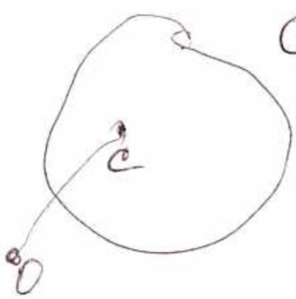
Excision for Singular Homology

Theorem: $B, A \subset X$ top pairs $X = A \cup B$ and A and B are open in X.

Then $\exists$ natural isomorphism

$H_\bullet(A, A \cap B) \simeq H_\bullet(X, B)$

In fact, it's given by $A \hookrightarrow X$.



$C \subset \mathbb{R}^N$ convex

$s_{n,c} : C_n(C) \to C_{n+1}(C)$

$T : \Delta^n \to C$

$s_{n,c}\left((1-t)z + t\vec{e}_{n+1}\right) := (1-t)\, T(z) + t \cdot c.$

$\Delta^{n+1} = \{(1-t)z + t\vec{e}_{n+1} \mid t \in I, z \in \Delta^n \subset \mathbb{R}^{n+1}\}$

$\vec{e}_{n+1} = (0, \dots, 0, 1)$

$$\mathbb{R}^{n+2} = \{(x_0, x_1, \dots, x_{n+1})\} \supset \mathbb{R}^{n+1} = \{(x_0, \dots, x_n)\}$$

Barycentric Subdivision

$$sd_{\Delta^n}(id_{\Delta^n}) := (-1)^n S_{n,\sigma}\, sd_{\Delta^n}(d(id_{\Delta^n}))$$

recursive definition of $sd_{\Delta^{n-1}}$

$\Delta^n \xrightarrow{id} \Delta^n \quad sd_{\Delta^n}: C_n(X) \to C_n(X) \quad sd_{\Delta^0} = id$

$sd_{\Delta^0}(id_{\Delta^0}) = id_{\Delta^0}$

where σ is the barycenter of $\Delta^n = \{(t_0, t_1, t_n) \mid \sum t_i = 1,\ t_i \geq 0\}$

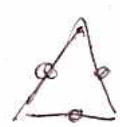
$$\sigma = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right)$$

$$d(\overset{0}{\bullet}\!-\!\!-\!\!-\!\overset{1}{\bullet}) = 1 - 0$$

$1\ 0 : \Delta^0 \to \Delta^1 = I$

$0 \longleftarrow \tfrac{1}{2} \quad \tfrac{1}{2} \longrightarrow 1$

$sd_{\Delta^1}(id_{\Delta^1}) = \overset{0}{-\!\circ\!\!\longrightarrow}{}^{1/2} + {}^{1/2}$



Δ^2

$$sd_X(T) = T_*(sd_{\Delta^n}(id_{\Delta^n}))$$

arbitrary

$T: \Delta^n \to X$

$T = T_*(id_{\Delta^n})$ $\quad \Delta^n \xrightarrow{id} \Delta^n \xrightarrow{T}$,,

Thus, sd_X ~~compact~~ $: C_n(X) \to C_n(X)$ is natural.

Lemma: 1. sd_X is a natural chainmap ($\partial sd_X = sd_X \partial$)

2. Every n-simplex of $sd_{\Delta^n}(id_{\Delta^n})$ is spanned by vertices $\hat{\sigma}_0, \dots, \hat{\sigma}_n$ for an strictly increasing collection of subfaces of Δ^n an $\hat{\sigma}_i$ is the barycenter of σ_i.

3. Let $\mathcal{W}$ be a collection of open subsets of X covering X, $T: \Delta^n \to X$. Then $\exists N: sd_X^N(T) = \sum_i m_i T_i$ so that each $T_i \subset W$ for some $W \in \mathcal{W}$

($\Delta^n \xrightarrow{T} X$

$\{T^{-1}(W) \mid W \in \mathcal{W}\}$ open covering of Δ^n

Lebesgue's Lemma: $\exists \varepsilon > 0$ such ~~that~~ that every subset of diameter $< \varepsilon$ is contained in one element of the covering)

4. $\forall m\ \exists$ a neutral chain homotopy
$S_X : id \simeq (sd_X)^m$

$$C_n \xrightarrow{id} C_n\ (sd_X)^n : C_n \to C_n$$

Construct it similar to how we constructed a homotopy for i_* and j_* in proving Homology Axiom.
(method of acyclic models: transferred this homotopy from Δ^n by $T_*(id_{\Delta^n}) = T$,
using $H_q(\Delta^n) = \begin{cases} G & q = 0 \\ 0 & q \neq 0 \end{cases}$
and knowing $id = (sd_{\Delta^n})^m_*$ in homology $H_0(\Delta^n)$.