Math 8307 Lecture 9 (M 07.02.2005)

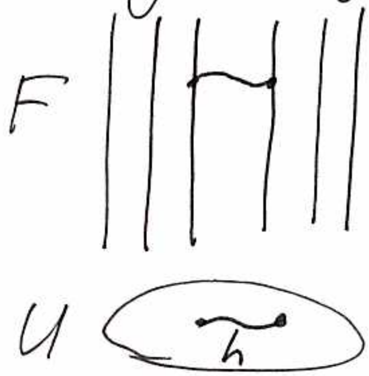
<u>Theorem</u>: If $p: E \to B$ is a map and $\mathcal{U}$ a numerable (loc. finite with fcts $h_u: B \to I$, $h_u^{-1}(0,1] = u$ for each $u \in \mathcal{U}$) open cover of B, then p has the HLP $\iff$ $p|_{p^{-1}(u)}: p^{-1}(u) \to u$ has the HLP for each $u \in \mathcal{U}$.

<u>Corollary</u>: If p is a fibre bundle over a paracompact base (in particular, a CW complex), then p is a fibration.

Indeed, locally a fiber bundle is $U \times F \to U$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & U \times F \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & U \end{tikzcd}$$

$$\tilde{h}(y,t) = (h(y,t), p_2 f(y))$$



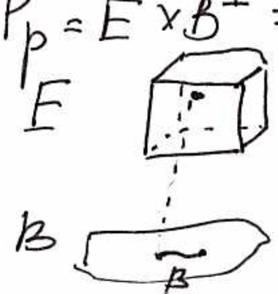
<u>Proof</u>: ($\Rightarrow$) obvious, because pullbacks (e.g., $p|_{p^{-1}(u)}$) of fibrations are fibrations.

($\Leftarrow$) Assume $p^{-1}(u) \xrightarrow{p} u$ are fibrations for all $u \in \mathcal{U}$

<u>Plan</u> We'll use the universal test diagram

$$\begin{tikzcd} E \arrow[d] & & E^I \arrow[ll, "p_0"'] \arrow[d] \\ & P_p \arrow[ul, "f"'] \arrow[ur, dashed, "\tilde{h}"] \arrow[dr, "h"] & \\ B & & B^I \arrow[ll, "p_0"] \end{tikzcd}$$

$$P_p = E \times_B B^I = \{(e,\beta) \mid p(e) = \beta(0)\}$$



and construct $\tilde{h}$ by patching the $\tilde{h}$'s for $p : p^{-1}(U) \to U$ for all $U \in \mathcal{U}$.

<u>Step 1</u> : <u>Put up the scaffolding of the patching argument</u> (it will be a numerable open cover of B^I)

$\mathcal{U}$ is numerable $\Rightarrow$ choose $\lambda_U : B \to I$

$\lambda_U^{-1}(0,1] = U \quad \forall\, U \in \mathcal{U}$.

$\forall$ finite ordered subset $T = \{U_1, \dots, U_n\}$ of the open cover $\mathcal{U}$ define C

and $\lambda_T : B^I \to I$ by

$$\lambda_T(\beta) = \inf\left\{ (\lambda_{U_i} \circ \beta)(t) \;\middle|\; t \in \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right], 1 \le i \le n \right\}$$

measures how far outside of the cover of U_i the path $\beta(t)$ goes over $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

Let $W_T = \lambda_T^{-1}(0,1] \subset B^I = \{\beta \in B^I \mid \beta(t) \in U_i \ \forall\, t \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \ \forall i\}$

Note that $\{W_T \mid T \text{ varies}\}$ is an open cover of B^I but no locally finite.

However $\{W_T \mid C(T) = n\}$ is locally finite for each n (use local finiteness of $\mathcal{U}$)

If $c(T) = n$, define

$\gamma_T : B^I \to I$

$$\gamma_T(\beta) = \max\{0, \lambda_T(\beta) - n \sum_{S:\, c(S)<n} \lambda_S(\beta)\} \in I$$

and define $V_T = \gamma_T^{-1}(0,1] = \{\beta \in B^I \mid \gamma_T(\beta) > 0\}$ $\subset W_T$

Then $\{V_T \mid T \text{ varies}\}$ is a locally finite cover of B^I

(See why for yourself)

~~Now~~ Step 2 (Patching)