Math 8307 Lecture 8 (F 04.02.2005)

Fibrations

Local Fibrations are fibrations

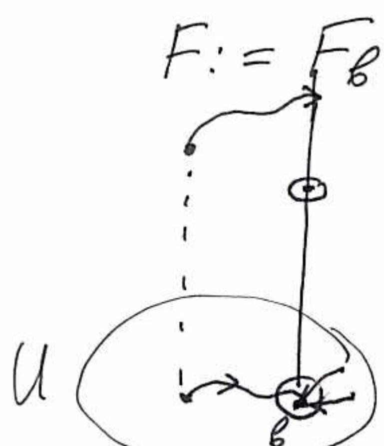
A covering space $E \to U$ over a contractible base U is isomorphic to

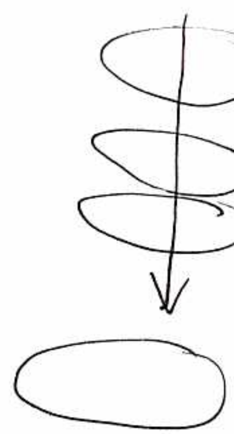
$$\begin{array}{c} U \times F \\ \downarrow \pi_1 \\ U \end{array}$$

with F a fixed discrete space, i.e.

$\exists$ homeomorp. $E \xrightarrow{\cong} U \times F$

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr] & U \times F \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$





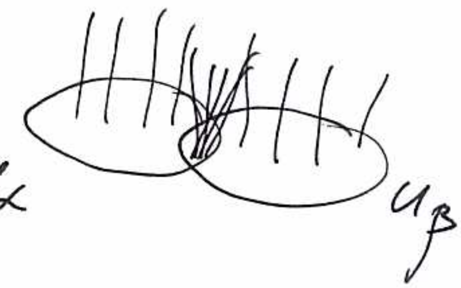
Def. A *fiber bundle* is a map $p : E \to B$, such that $\exists$ open cover $\{U_\alpha\}$ of B and homeomorphism $\varphi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $p \circ \varphi_\alpha = \pi_1$

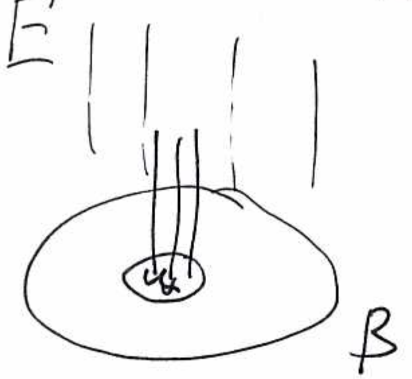
$$\begin{tikzcd} U_\alpha \times F \arrow[r, "\varphi_\alpha"] \arrow[dr, "\pi_1"'] & p^{-1}(U_\alpha) \arrow[d, "p"] \\ & U_\alpha \end{tikzcd}$$

for a fixed space F

From this you get $\varphi_\beta^{-1}\varphi_\alpha$ <u>gluing functions</u>

$$
\begin{array}{ccc}
(U_\alpha \cap U_\beta)\times F & \xrightarrow{\varphi_\beta^{-1}\varphi_\alpha} & (U_\alpha \cap U_\beta)\times F \\
& \searrow \quad \swarrow & \\
& U_\alpha \cap U_\beta &
\end{array}
$$





<u>Ex.</u> $\mathbb{C}^{n+1}\setminus\{0\} \longrightarrow S^{2n+1}$

$\mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{C}P^n$

$S^{2n+1} \longrightarrow \mathbb{C}P^n$

$\mathbb{R}^{n+1}\setminus\{0\} \longrightarrow S^n$

$\mathbb{R}^{n+1}\setminus\{0\} \longrightarrow \mathbb{R}P^n$

$S^n \longrightarrow \mathbb{R}P^n$ double covering map $\Rightarrow$ fiber bundle

Proof of the fact $S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$ is a fiber bundle

$U_0, U_1, \dots, U_n$

$U_i = \{[z_0 : z_1 : \dots : z_n] \mid z_i \neq 0\}$

$F = S^1 \qquad \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0}\right)$

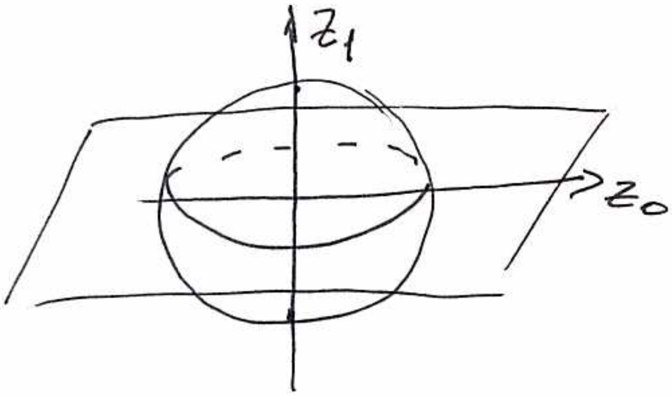
$U_0 \cong \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n\} = \mathbb{C}^n$

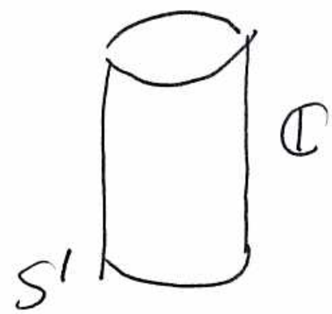
$p^{-1}(U_0) = \{(z_0, z_1, \dots, z_n) \in S^{2n+1} \mid z_0 \neq 0\}$

$$\varphi_0 : U_0 \times F \to p^{-1}(U_0)$$

$$(x_1, x_2, \dots, x_n, z) \longmapsto (\lambda z, \lambda x_1 z, \dots, \lambda x_n z), \text{ where}$$

$$\lambda = \left(1 + |x_1|^2 + \dots + |x_n|^2\right)^{-1/2}$$



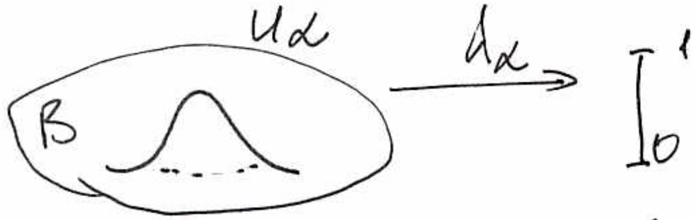


Back to fibrations

<u>Def.</u> A <u>numerable open cover</u> of a space B is an open cover $\{U_\alpha\}$ satisfying

(1) $\forall \alpha \; \exists \lambda_\alpha : B \to I$
$\lambda_2^{-1}((0,1]) = U_\alpha$

(2) $\{U_\alpha\}$ is locally finite, i.e., $\forall b \in B \; \exists$ neighbhd V of b such that $U_\alpha \cap V \neq \emptyset$ only for $< \infty$ α's.



<u>Fact</u> Any open cover of a paracompact space (with our condition of weak Hausdorfness) admits a numerable refinement.

<u>Theorem</u> $p: E \to B$ $\{U_\alpha\}$ numerable cover of B. Then p is a fibration $\Leftrightarrow$ $p: p^{-1}(U_\alpha) \to U_\alpha$ is a fibration $\forall \alpha$.

<u>Corrolary</u>: Every fiber bundle is a fibration.