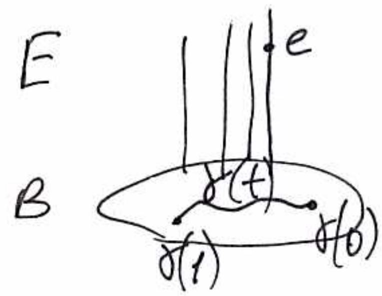
Math 8307 Lecture 7 (02.02.2005)

Fibrations

Mapping Path Space

For $p: E \to B$, $\exists$ a universal test diagram for p to be a fibration with Y: $Pp := E \times_B B^I$
$= \{(e, \gamma(t)) \mid p(e) = \gamma(0)\} \subset E \times B^I$

$$\begin{tikzcd} Pp \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B^I \arrow[d, "p_0"] \\ E \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} \gamma(t) \arrow[d] \\ \gamma(0) \end{tikzcd}$$



Pp called <u>mapping path space</u>

Suppose p satisfies the HLP just for one test diagram, namely the following:

$$\begin{tikzcd} E \arrow[d, "p"'] & & E^I \arrow[ll, "p_0"'] \arrow[d, "p^I"] \\ & Pp \arrow[ul, "f"] \arrow[ur, dashed, "\tilde{h}"] \arrow[dr, "h"] \arrow[l, phantom, "\text{pullback square}"] & \\ B & & B^I \arrow[ll, "p_0"] \end{tikzcd}$$

Then p is a fibration, because $\forall$ test diagram

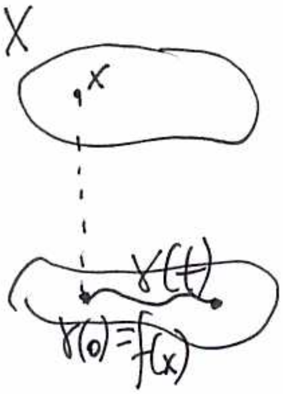
$$\begin{tikzcd} E \arrow[ddd, "p"'] & & & E^I \arrow[lll, "p_0"'] \arrow[ddd, "p^I"] \\ & & Pp \arrow[ull] \arrow[ur] \arrow[ddr] & \\ & Y \arrow[uul, "f"] \arrow[ur, "\exists!"] \arrow[drr, "h"] & & \\ B & & & B^I \arrow[lll, "p_0"] \end{tikzcd}$$

Replacing Maps by fibrations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "\nu"'] & & Y \\ & Pf \arrow[ur, "p"'] & \end{tikzcd}$$

given f, construct Pf, ν, p:

$$Pf = X \times_Y Y^I = \{(x,\gamma) \mid \gamma(0) = f(x)\}$$



$$\subset X \times Y$$

$$\nu(x) = (x, c_{f(x)})$$ ← constant path

$$p(x,\gamma) := \gamma(1)$$

<u>Claim</u>: ν is a h.e. and p satisfies the HLP.
In particular, p will be a fibration, if $f(X)$ intersects with each path component of Y (or $\forall y \in Y\ \exists$ path $\gamma(t)$ in Y and $x \in X$: $f(x) = \gamma(0)$, $\gamma(1) = y$), e.g., if f is surjective.

<u>Proof of claim</u>.

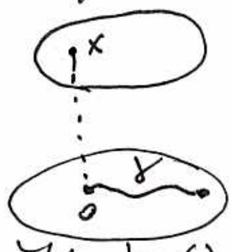
1. <u>ν is a h.e</u>

$\pi: Pf \to X$ the projection

$\pi \circ \nu = id_X$

$\nu \circ \pi \simeq id_{Pf}$

via $h: Pf \times I \to Pf$



$$h(x,\gamma)(t) = (x, \gamma_t) \text{ with } \gamma_t(s) = \gamma((1-t)s)$$

2. Check the HLP for $p : P_f \to Y$

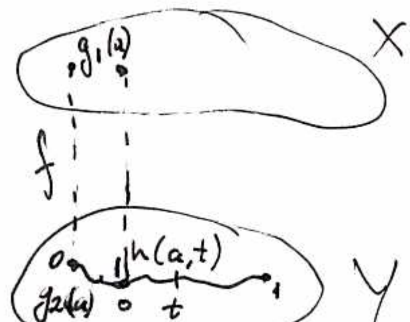
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & P_f \arrow[d, "p"] \\ A\times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd}$$

Define $\tilde{h}(a,t) = (g_1(a), j(a,t))$

where $g(a) = (g_1(a), g_2(a)) \in P_f \subset X \times Y^I$

and $j(a,t)(s) = \begin{cases} g_2(a)(s(1+t)) & 0 \le s \le \frac{1}{1+t} \\ h(a, s(1+t)-1) & \frac{1}{1+t} \le s \le 1 \end{cases}$

$\forall a \in A$



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