Math 8307 Lecture 6 (M 31.01.2005)

Fibrations

Extra to cofibrations:

$\tilde{H}_\cdot(X/A) \cong H_\cdot(X,A)$

if $A \hookrightarrow X$ is a cofibration.

This implies that if $A \hookrightarrow X$ is a cofibration, then we have a LES

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots$$

which comes from

(compare with LES ~~for~~ of a triple

$$\cdots \to H_n(A,*) \to H_n(X,*) \to H_n(X,A) \to H_{n-1}(A,*) \to \cdots$$

<u>Def.</u>: $p: E \to B$ a surjective map is a (Hurewicz) fibration if $\forall$ diagram satisfies the HLP (homotopy lifting property): (sometimes homotopy covering property)

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

E ← total space

B ← base of the fibration

<u>Remark</u>: A Serre fibration is a more general notion, when one uses I^n for test spaces Y.

Test diagram for cofibrations

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{h}"] & Y \end{tikzcd}$$

Let's reverse the arrows on the diagram for fibration (and replace E by X, B by A)

$$\begin{tikzcd} Y & \cancel{E}\, X \arrow[l] \arrow[dl, dashed] \\ Y^I \arrow[u] & \cancel{B}\, A \arrow[l] \arrow[u] \end{tikzcd}$$

$V \longmapsto V^* = Hom_k(V, k)$

$X \longmapsto Map(X, X_0)$

$Y \longmapsto Map(Y, X_0) = \tilde{Y} = X_0^{\,Y}$

$Y \times I \to Map(Y \times I, X_0) = \tilde{Y}^I \qquad Map(I, X_0^Y) = \tilde{Y}^I$

$\qquad\qquad \| \qquad\qquad =$

$\qquad Map(Y, X_0^I)$

Another equivalent form of the test diagram is:

$$\begin{tikzcd} E & & E^I \arrow[ll, "p_0"'] \arrow[dd, "p^I"] \\ & Y \arrow[ul, "f"] \arrow[ur, dashed, "\tilde{h}"] \arrow[dr, "h"] & \\ B \arrow[from=uu, "p"'] & & B^I \arrow[ll, "p_0"] \end{tikzcd}$$

$p_0(\gamma) = \gamma(0)$

Prop

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

pullback (dual to pushout)

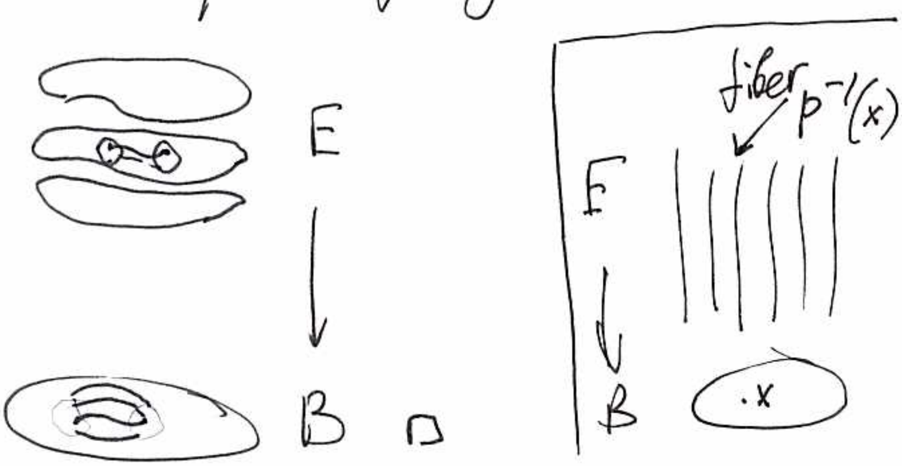
$A \times_B E = \{(a,e) \mid g(a) = p(e)\} \subset A \times E$

If p is a fibration, then $A \times_B E \to A$ is also a fibration

(When you have a fibration it is also a fibration locally).

every fiber bundle is a fibration

Prop. A covering map $p : E \to B$ is a fibration with unique lifting $\tilde{h}$ in any test diagram



Lemma: If $A \to X$ is a cofibration and B a space, then the induced map $B^X \to B^A$ is a fibration.

$$\begin{tikzcd} B^X \arrow[d] & & (B^X)^I \arrow[ll, "p_0"'] \arrow[d] \\ & Y \arrow[ul] \arrow[ur, dashed, "\tilde{h}"] \arrow[dr] & \\ B^A & & (B^A)^I \arrow[ll, "p_0"] \end{tikzcd}$$

$\mathrm{Map}(Y, B^X)$

$\mathrm{Map}(Y \times X, B)$

$\mathrm{Map}(X, B^Y)$

$Y \times X$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & X \times I \arrow[dl, dashed] \\ & B^Y \\ A \arrow[r] & A \times I \arrow[ul] \arrow[uu] \end{tikzcd}$$