Math 8307 Lecture 5 (F 28.01.2005)

Cofiber Homotopy Equivalence

A space X under A is a map

$$A \to X$$

A map under A is a comm. diagram

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

i.e. $f(i(a)) = j(a)$ $\forall a$

A homotopy under A is
a homotopy $h: X \times I \to Y$
such that $h(i(a), t) = j(a)$ $\forall t \in I$ $a \in A$

Then we write $f \sim f'$ rel A for $f(x) = h(x,0)$,
$f'(x) = h(x,1)$

Thus, we get the notion of
a h. eq. under A

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr, shift left, "g"] & & Y \arrow[ll, shift left, "f"] \end{tikzcd}$$

$fg \sim id_X$ rel A

$gf \sim id_Y$ rel A

f, g are under A

A cofiber homotopy equivalence
is a h. equiv. under A.

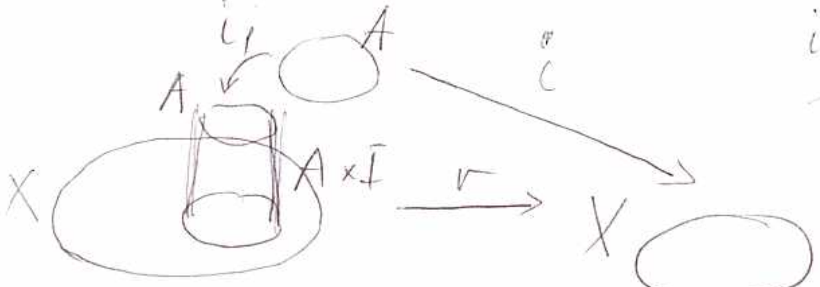
Prop. Let $i : A \to X$, $j : A \to Y$ be cofibrations and $f : X \to Y$ a map under A. Suppose f is a h. equivalence. Then f is a cofiber h. equivalence.

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Ex.

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "i"] & \\ M_i \arrow[rr, "r"] & & X \end{tikzcd}$$

For a cofibration $i : A \to X$, M_i and the diagram



$i_0 r$: collapses of M_i to its bottom X

r is a homotopy equivalence with a homotopy inverse $i_0 : X \to M_i$, $r i_0 = id_X$, $i_0 r \sim id_{M_i}$

i_0 is not a map under A

because $Im(i_0 i) \subset A \times \{0\}$ while $Im(i_1) \subset A \times \{1\}$

But Prop. $\Rightarrow$ $i_0 \sim$ a map which is a h. inverse of r under A.

Indeed, Prop. $\Rightarrow \exists\, g$ under A which is a h. inverse of r under A.

But i_0 and g are h. inverse of r $\Rightarrow$ $i_0 \sim g$ (by the uniqueness of an inverse in a category) not under A

Prop. Assume

$$\begin{array}{ccc} A & \xrightarrow{d} & B \\ {\scriptstyle i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

with i, j cofibrations and d, f h.equivalences.
Then $(f, d) : (X, A) \to (Y, B)$
is a h.eq. of pairs.

Theorem If $A \hookrightarrow X$ is a cofibration, then
$H_\bullet(X, A) \simeq \tilde{H}_\bullet(X/A)$

Proof: (X, A) is an NDR

$u : X \to I \quad u^{-1}(0) = A$

$h : X \times I \to X \quad h(x, 1) \in A$ for $x \in u^{-1}([0, 1))$

$h(a, t) = a \quad \forall a, t$

$h(x, 0) = x \quad \forall x$

$W := u^{-1}([0,1)) \subset X$

h is then a deformation retraction of W onto A and also induces a d. retraction of W/A onto $A/A = \{*\}$

$\tilde{H}_\bullet(X/A) \xrightarrow{\simeq} H_\bullet(X/A, W/A)$

$\parallel \quad$ LES

$H_\bullet(X/A, A/A)$

LES:

$$\to H_n(W/A, A/A) \to H_n(X/A, A/A) \xrightarrow{\cong} H_n(X/A, W/A) \to$$

$$\overset{\shortparallel}{0}\ \forall n$$

$$H_\cdot(X/A, W/A) \underset{\uparrow \text{ excision}}{\xrightarrow{\cong}} H_\cdot(X \setminus A, W \setminus A)$$

$$\underset{\uparrow \text{ excision}}{\cong} H_\cdot(X, W) \underset{\uparrow \text{ LES}}{\xrightarrow{\cong}} H_\cdot(X, A)$$

$\square$

<u>Theorem</u> If A is a subcomplex of a CW cx X, then $A \hookrightarrow X$ is a cofibration.

<u>Proof</u>: prove HEP by induction on skeleton, doing one cell at a time. $\square$