Math 8307 Lecture 4 (W 26.01.05)

A local Test for a cofibration

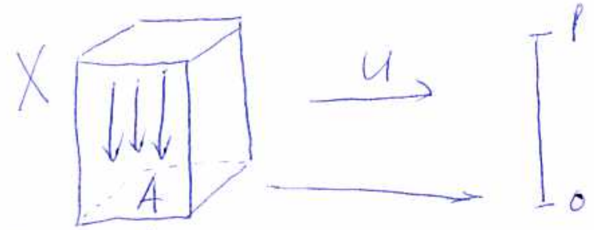
(A, X) pair: $A \hookrightarrow X$ a subspace of a space

<u>Def. 1</u> A pair (X, A) is an <u>NDR-pair</u> (neighbourhood deformation retract) if $\exists u : X \to I : u^{-1}(0) = A$ and a homotopy $h : X \times I \to X$

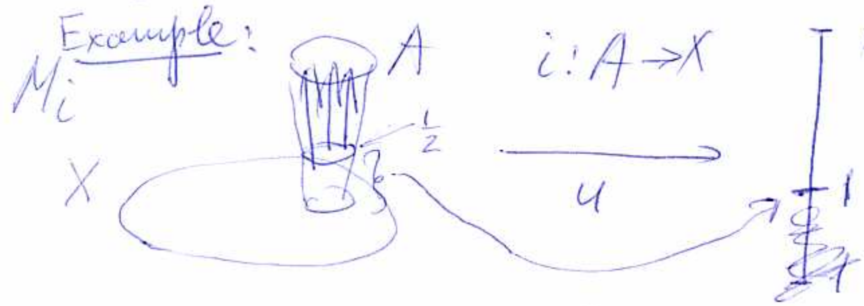
$h_0 = id$, $h(a,t) = a \quad \forall a \in A, t \in I$

$h(x,1) \in A$ if $u(x) < 1$.

2. (X, A) is a DR-pair if $u(x) < 1$ $\forall x \in X$ in addition to the above.



Example:



A is an NDR of M_i (i.e. (M_i, A) is an NDR-pair)

We'll see this implies $A \hookrightarrow M_i$ is a cofibration.

<u>Lemma</u> Suppose $(X, A), (Y, B)$ are NDR-pairs via $(u, h), (v, j)$, resp. Then $(X \times Y, X \times B \cup A \times Y)$ is an NDR-pair via (w, k) with $w(x,y) = \min(u(x), v(y))$

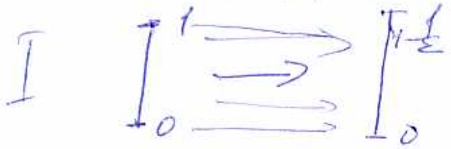
$$k(x,y,t) = \begin{cases} \left(h(x,t),\ j\left(y,\ t u(x)/v(y)\right)\right) & \text{if } v(y) \geq u(x) \\ \left(h\left(x,\ t v(y)/u(x)\right),\ j(y,t)\right) & \text{if } u(x) \geq v(y) \end{cases}$$

Idea of proof: First, check k is well defined and continious: when $v(y)=0$ and $v(y) \geq u(x)$, then $u(x)=0$

$\Rightarrow x \in A,\ y \in B$. Then $k(x,y,t)=(x,y)$ and this also implies continuity of k $\square$

Lemma: If in the previous Lemma (X,A) or (Y,B) is a DR-pair, then $(X \times Y,\ X \times B \cup A \times Y)$ is also a DR-pair (via the same maps).

Example. $(I, \{0\})$ is a DR-pair.



If (X,A) is an NDR-pair, then $(X \times I,\ \underbrace{X \times \{0\} \cup A \times I}_{M_i})$ is a DR-pair $\quad i: A \hookrightarrow X$

Theorem. Let $A \subset X$ be a closed subspace. Then TFAE the following are equivalent.

wlog without loss of generality

(i) (X,A) is an NDR-pair

(ii) $(X \times I,\ \underbrace{X \times \{0\} \cup A \times I}_{M_i})$ is a DR-pair

(iii) $M_i = X\times\{0\}\cup A\times I$ is a retract of $X\times I$,

(iv) $i: A \hookrightarrow X$ is a cofibration.

<u>Proof</u>: (i) $\Rightarrow$ (ii) by Lemma

(ii) $\Rightarrow$ (iii) trivially.

(iii) $\Leftrightarrow$ (iv) done last time (M_i is a universal test space for i to be a cofibration).

(iii) $\Rightarrow$ i Suppose $r: X\times I \to M_i$ is a retraction.

$$(\underbrace{M_i \hookrightarrow X\times I \xrightarrow{r} M_i}_{id})$$

Let $\pi_1: X\times I \to X$, $\pi_2: X\times I \to I$

Define ~~$u(x)$~~ $u: X \to I$ by

$$u(x) = \sup\{t - \pi_2 r(x,t) \mid t\in I\} \in I$$

and $h(x,t) = \pi_1 r(x,t)$.

Then $u^{-1}(0) = A$ because

$u(x) = 0 \Rightarrow r(x,t)\in A\times I$ for $t>0$.

because $\forall t$, $t - \pi_2 r(x,t) \leq 0$

$\pi_2 r(x,t) \geq t > 0$.

so $\pi_2 r(x,t) > 0$

$$\Rightarrow r(x,t)\in A\times I \hookrightarrow M_i \quad \forall t>0$$

$\Rightarrow r(x,t)\in A\times I$ for $t=0$ as well, because $A\times I \hookrightarrow X\times I$ is closed.

but $r(x,0) = (x,0)$, because $(x,0) \in M_i$

$\Rightarrow x \in A$.

The rest is simple $\Box$.