Math 8307 Lecture 3 (M. 24. 01. 2005)

Mapping Cylinders & Cofibrations

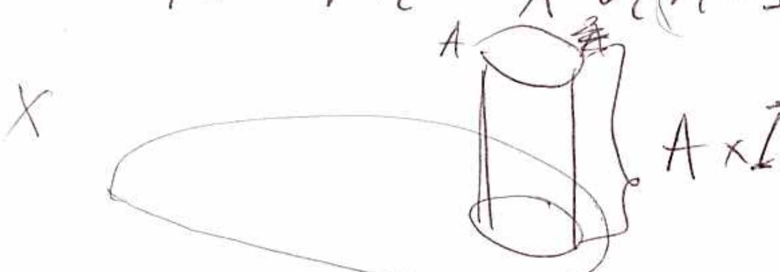
Reminder: $A \xrightarrow{i} X$ is a cofibration if it satisfies the HEP $\forall$ spaces Y!

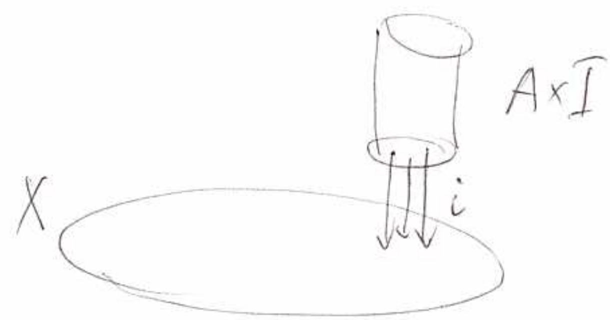
$$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A\times I \arrow[dl, "h"'] \arrow[dd, "i\times id"] \\ & Y & \\ X \arrow[ur, "f"] \arrow[rr, "i_0"] & & X\times I \arrow[ul, dashed, "\tilde h"'] \end{tikzcd}$$

Major example will be a CW-pair (X, A) (a CW cx X and a subcomplex A)

It turns out, $\exists$ a universal test diagram with

$$Y := M_i = X \cup_i (A\times I)$$





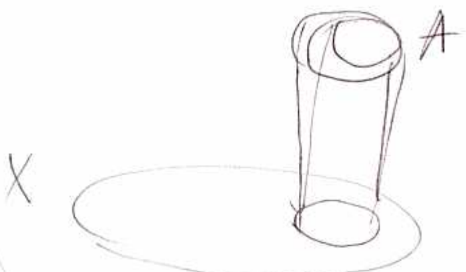
$$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"'] \arrow[dr, phantom, "\text{pushout square}"] & & A\times I \arrow[dl] \arrow[dd, "i\times id"] \\ & M_i & \\ X \arrow[ur] \arrow[rr, "i_0"] & & X\times I \arrow[ul, dashed, "r"'] \end{tikzcd}$$

Suppose the universal test diagram satisfies HEP, i.e., $\exists\, r$ making the diagram commutative

$$\begin{tikzcd}
A \arrow[rrr, "i_0"] \arrow[dd, "i"'] \arrow[drr, phantom, "\text{pushout}"] & & & A\times I \arrow[dll] \arrow[dl, "h"] \arrow[dd, "i\times id"] \\
& M_i \arrow[r] & Y & \\
X \arrow[ur] \arrow[urr, "f"] \arrow[rrr, "i_0"'] & & & X\times I \arrow[ull, dashed, "r"] \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

pushout square $A \to A\times I \to M_i \leftarrow X$
$\Rightarrow$ (univers. prop)
$\Rightarrow$ there is a unique map from M_i to Y.
Then $\tilde{h}$ because triangles commutative.

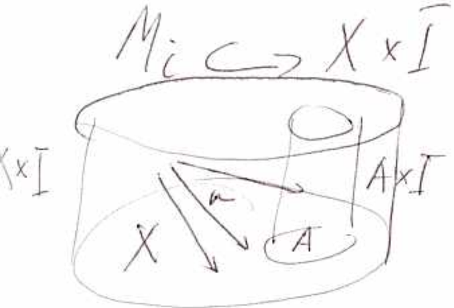
<u>Prop</u>. $\overbrace{i: A \to X}^{\text{An inclusion}}$ is a cofibration if and only if M_i is a retract of $X\times I$.



<u>Remark</u> If such r exists then we have $rj = id$ for $j: M_i \to X\times I$ defined as $\underset{x\in X}{j(x)} = (x,0)$, $\underset{a\in A}{j(a,t)} = (i(a),t)$

$$j(a,0) = (i(a),0).$$

Hint to HW problem:
If $i: A \to X$ is a cofibration then it's an inclusion with closed image.
Hint: use the remark.



Prop. follows from the above observation.

Replacing maps with cofibrations

$f: X \to Y$

$\rightsquigarrow$

$f: X \xrightarrow{\text{cof.}} \tilde{Y} \xrightarrow{\text{h.e.}} Y$

$f: X \overset{i_1}{\hookrightarrow} Mf \xrightarrow{r} Y$

$r(y) = y$
$r(x,t) = f(x)$

r is a h.e., because $r i_0 = id_Y$ and $i_0 r$ is h.e. to id_{Mf}: $i_0 r \sim id_{Mf}$ via a homotopy:

$h: Mf \times I \to Mf$

between id and r:

$h(y,t) = y$ for $y \in Y$

$h((x,s),t) = (x,(1-t)s)$ for $(x,s) \in X\times I$, $s>0$.

i_1 is a cofibration

i_1 is a cofibration because it satisfies the HEP $\forall Y$ (which may be checked directly) or may be seen from noticing that it's a neighbourhood deformation retract (NDR)