Math 8307 Lecture 2 (21.01.2005)

Reminder: Compact = compact & Hausdorff

(top) spaces = top. spaces which are compactly generated

We apply the functor $(-)_c$ to

$X \times Y = (X \times Y)_c$

$X^Y := Map(Y, X) = (Map(Y, X))_c$

and whenever we make a construction that takes us out of the cat. of c.g. spaces.

Cofibrations

LES's are important, e.g. the LES in homology

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots$$

for a subcomplex A of a CW complex X.

The Cofiber sequence is in a sense a lift of a LES to the category of spaces:

$$A \to X \to A/X \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \cdots$$

which is true, if $A \to X$ is a cofibration

And also a cofibration generalizes a subcomplex.

Fibration and fiber sequences are Hilton-Eckman dual to cofibr's & cofiber sequences

Def. A map $i : A \to X$ is a cofibration if it $(\forall Y)$ satisfies the HEP (homotopy extension property) for any Y, h, and f.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A\times I \arrow[dl, "h"'] \arrow[d, "i\times id"] \\
Y & \\
X \arrow[u, "f"] \arrow[r, "i_0"] & X\times I \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

$i_0(a) = (a,0)$

commut-ty of this triagle means that $\tilde{h}$ extends h

comm-ty means $\tilde{h}$ satisfies the initial condition

An equivalently diagram:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I = Map(I, Y) \arrow[d, "p_0"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{h}"] & Y
\end{tikzcd}$$

$p_0(\gamma) = \gamma(0)$

HW Proplem. A cofibration is always an inclusion with closed image
(There will be a hint when we look at the mapping cylinder)

Digression

Pushouts:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d] \\
B \arrow[r] & B \coprod_A X
\end{tikzcd}$$

union A, B identify image in A = image in B

$= (B \coprod X) / (i(a) \sim g(a) \ \forall a \in A)$

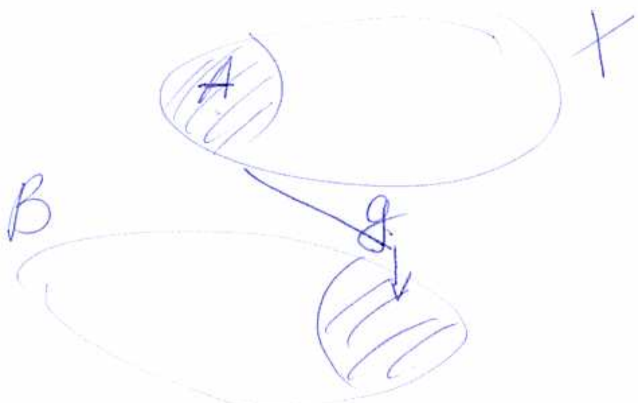
disjoint union

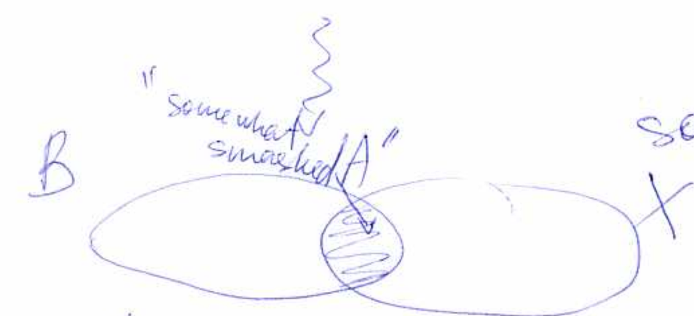
it is a colimit in cat. sense and has universality property

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & B \coprod_A X \arrow[dr, dashed, "!"] & \\
 & & Y
\end{tikzcd}
$$

∀ the large square (with Y) is commut. there is a unique map making the diagram commutative
universality property of the pushout

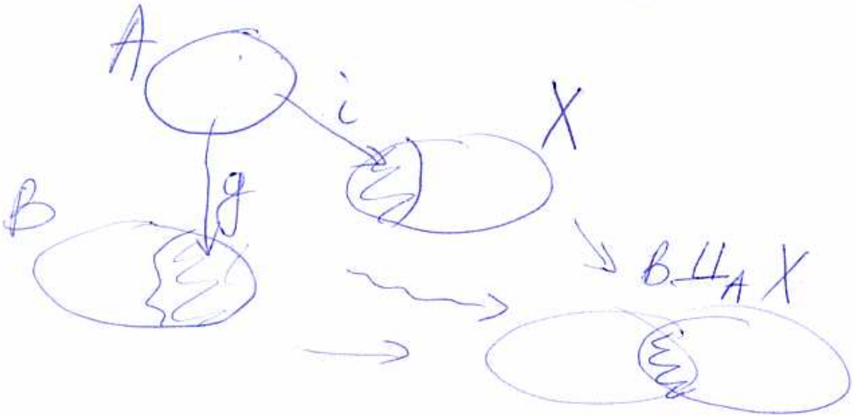
When $i: A \hookrightarrow X$ is an inclusion, the pushout is denoted by $B \cup_g X$





some of the points of A can be glued together.

<u>Lemma</u> The pushout of a cofibration is a cofibration is a cofibration:
if $A \xrightarrow{i} X$ is a cofibration and $A \xrightarrow{g} B$ is a conts. map, then $B \to B \cup_g X$ is also a cofibration.



$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d] \\ B \arrow[r] & B \underset{A}{\sqcup} X \end{tikzcd}$$

Proof: $(B \cup_g X) \times I = (B \times I) \cup_{g \times id} (X \times I)$

$$\begin{tikzcd} A \arrow[rrrr, "i_0"] \arrow[dr, "g"] \arrow[dddd, "i"'] & & & & A \times I \arrow[dl, "g \times id"'] \arrow[dddd, "i \times id"] \\ & B \arrow[rr, "i_0"] \arrow[dd] & & B \times I \arrow[dl, "h"'] \arrow[dd, "\text{pushout}"] & \\ & & Y & & \\ & B \cup_g X \arrow[ur, "f"] \arrow[rr, "i_0"] & & (B \cup_g X) \times I \arrow[ul, "\tilde h"'] & \\ X \arrow[ur] \arrow[rrrr, "i_0"] & & & & X \times I \arrow[ul] \arrow[uull, dashed, bend right, "\bar h"'] \end{tikzcd}$$

pushout diagram

$\bar h$ exists because of HEP for the large sq.

$\bar h \longrightarrow \tilde h$

$\tilde h$ exists because of $\bar h$ and universal property for pushout square