Math 8307 Lecture 1

Homotopic theory.

Compactly generated spaces

Pathologies of arbitrary topological spaces:

$\mathrm{Map}(X, Y) = Y^X$

$Z^{X \times Y} \cong (Z^Y)^X$ adjointness bijection

$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(X, Z))$

These are not continious bijections (or homeomorphisms) for general spaces X, Y, Z and the standard topologies on Z^Y and $X \times Y$

If you consider X, Y CW complexes

top. space $X \times Y$ is not a CW complex in general

counterexample

X with uncountably many intervals

Y with countably many intervals.

Compact = compact & Hausdorff
à la Bourbaki

weak Hausdorff := $g(K)$ is closed in X $\forall\, g : K \to X$ with K compact

Evry compact (in the old sense) subset is closed $\overset{\not\Leftarrow}{\Longrightarrow}$ w. H

Evry compact (in the Bourbaki sense) subset is closed $\Leftarrow$ w. H.

If X is w.H. and $g: K \to X$, K compact, then $g(K)$ is Hausdorff and therefore compact in the B. sense.

$T_1 <$ w.H. $<$ Hausdorff
points are closed "not much"

$A \subset X$ compactly closed, if $g^{-1}(A)$ is closed in K for $\forall g: K \to X$ with K compact

A closed $\Rightarrow$ compactly closed

When X is w.H. , $A \subset X$ is compactly closed if and only if $(\Leftrightarrow)$ $\forall K \subset X$ compact $K \cap A$ is closed in K)

X is a <u>k-space</u> $\Leftrightarrow$ every compactly closed subspace is closed.

X is <u>compactly generated</u> $\Leftrightarrow$ it's weak H. and a k-space

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Facts about c.g. spaces

Locally compact $\Rightarrow$ compactly generated.
$\hookrightarrow$ for each point there is a closed neibourhood.

w.H. + 1st axiom of countability (each point has a countable neighbhd basis) $\Rightarrow$ c.g.

If X is c.g. and Y arbitrary, then $f: X \to Y$ is continuous $\Leftrightarrow$: $f|_K : K \to Y$ is conts. for any compact subspace $K \subset X$.

<u>Main trick</u>: make any space X a k-space by giving it a new topology, in which $A \subset X$ is closed $\Leftrightarrow$ A is compactly closed in the original topology of X.

The resulting new space is X_c

Obviously $id: X_c \to X$ is continuous

~~Facts about the trick:~~

If X is w.H., then so is X_c, hence X_c is c.g.

X and X_c have exactly the same compact subsets. (again X is w.H.)

To all constructions of new topological spaces we'll be adding from now on by default the functor $(-)_c$, e.g. $X \times Y$ will (starting from next time) be $(X \times Y)_c$

If X is locally compact and Y is c.g., then $(X \times Y)_c = X \times Y$

X is H. $\Leftrightarrow \Delta \subset X \times X$ is closed

If X is a k-space, then X is w.H $\Leftrightarrow$
$\Leftrightarrow \Delta \subset (X \times X)_c$ is closed.

X CW complex $\Rightarrow$ X is c.g.
because it is H. (therefore w.H.) and $A \subset X$ is closed if and only if (iff) $A \cap \bar{e}_\alpha$ is closed

If X and Y are c.g. and $A \subset X$ is closed, $f: A \to Y$ is continuous, then the pushout $Y \cup_f X$ is compactly generated

If $X_i \hookrightarrow X_{i+1}$ and X_i's are compactly generated for $i \geq 1$, then $\operatorname{colim} X_i := \bigcup_{i \geq 1} X_i$ is c.g.

$Y^X := \mathcal{k}(Y^X)_c$, applied to Y^X is the standard, compact-open topology, in which a basis of ~~th~~ topology is given $\bigcap_{<\infty} \{ f: X \to Y \mid f(K) \subset U \}$ for given compact K in X and open U in Y.

Then for X, Y, Z c.g.

$$Z^{X \times Y} \cong (Z^Y)^X$$

homeomorphism